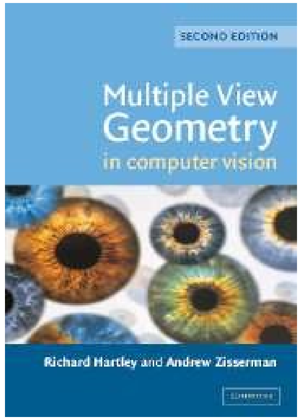
Crash-Course: The Oriented Projective Geometry of Multiple Views

Everything you need to know to write code for X-ray geometries and then some.

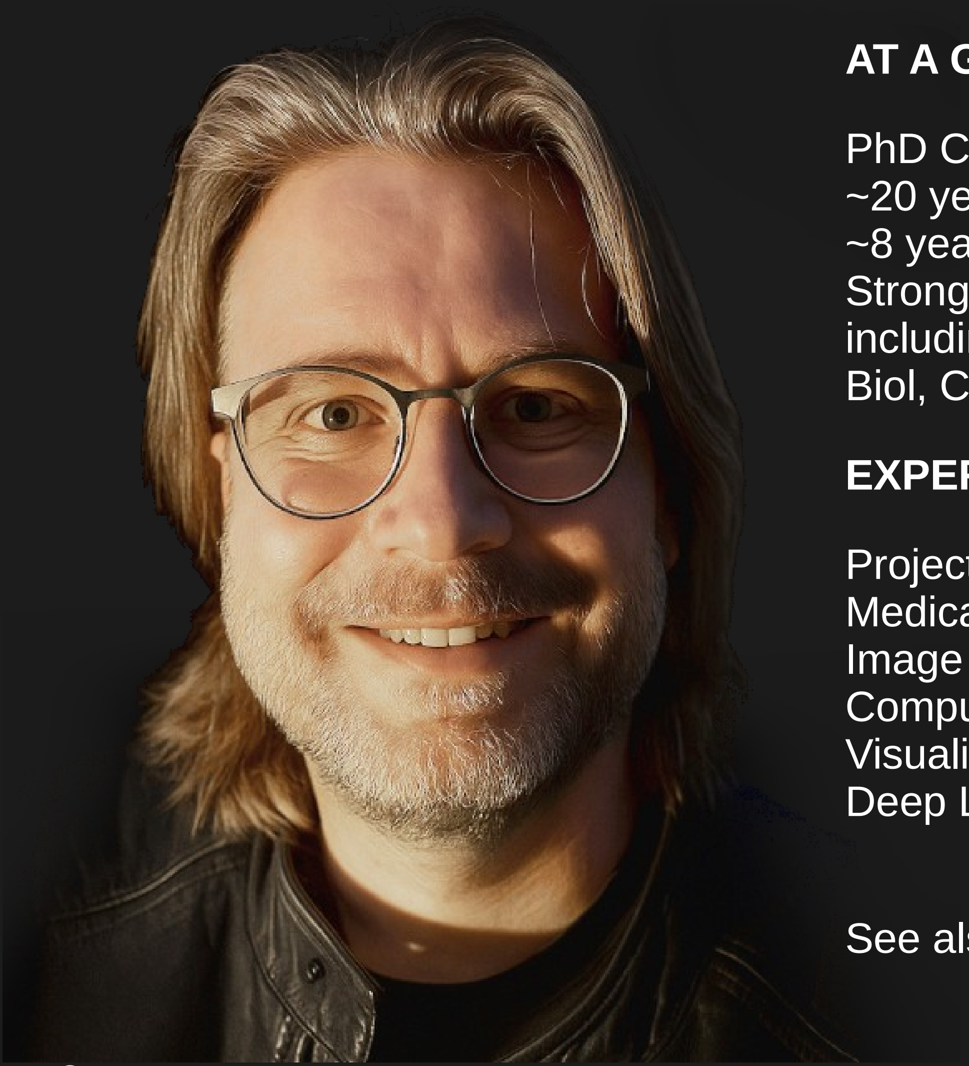


Book Recommendation
Multiple View Geometry in Computer Vision

Richard Hartley and Andrew Zisserman,
Cambridge University Press, March 2004.

https://www.robots.ox.ac.uk/~vgg/hzbook/

About the speaker



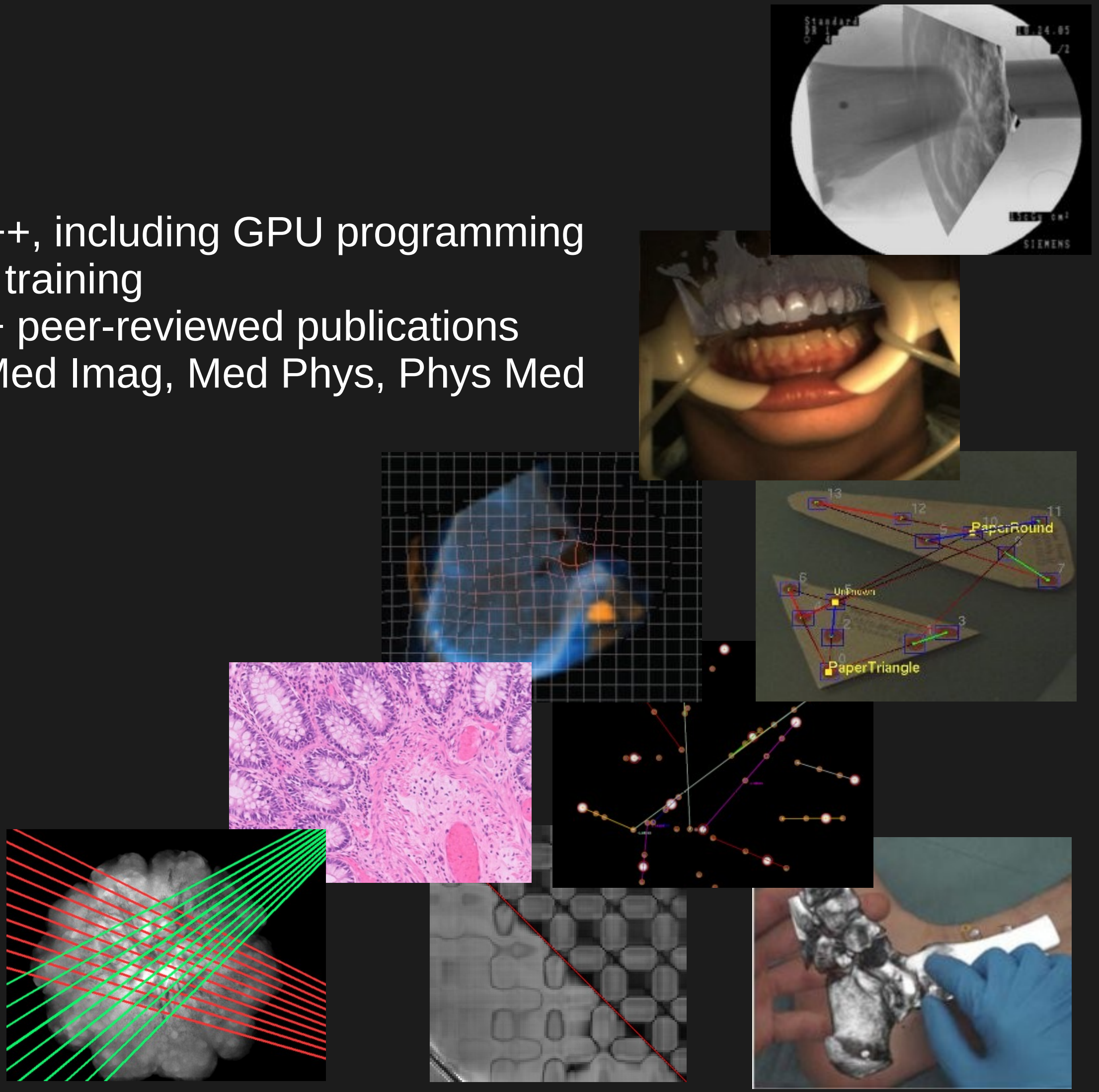
AT A GLANCE

PhD Computer Science
~20 years of experience in C/C++, including GPU programming
~8 years of Python, including AI training
Strong publication history of 30+ peer-reviewed publications including MICCAI, IEEE Trans Med Imag, Med Phys, Phys Med Biol, CT-Meeting

EXPERTISE

Projective Geometry
Medical Image Processing
Image Reconstruction
Computer Vision
Visualization & AR
Deep Learning

See also: aaichert.de



About the Seminar

Our goals and how to reach them.

01 **This is a seminar about my PhD thesis**

02 **Target audience is Msc level in math, computer science, electrical engeneering or related subjects with some level of programming experience.**

03 **The seminar teaches theoretical concepts. Each chapter is followed by a break with optional practical excercises in python (and some C++)**

04 **Practical excercises motivate relevance and show-case applications.**

05 **Please ask questions as they arise.**

06 **Slow me down to the speed that you can follow.**

07 **I’d rather skip some sub-topics than to overexhaust the audience.**

Crash-Course: The Oriented Projective Geometry of Multiple Views

Everything you need to know to write code for X-ray geometries and then some.

09:30	Projective Geometry of Two-Space
10:30	Projective Geometry of Three-Space
11:30	Practical: Computing With LInes
13:00	Anatomy of the Projection Matrix
14:30	Practical: Visualize Source-Detector Geometry
15:30	Summary, Outlook, Discussion

Julius Plücker
(16 June 1801 – 22 May 1868)

Felix Klein
(25 April 1849 – 22 June 1925)

Projective Geometry of Two-Space

01 Algebraic Description of 2D Lines and their Homogeneous Representation

02 Homogeneous Representatoin of Points and Duality

03 Points at Infinity and Direction

04 Incedence, Orthogonality and Duality: Join and Meet

05 The Plücker Matrix in Two-Space

06 Ontology of Linear Transformations and Their Invariants

Practical: Setup Jupyter, Computing with 2D Points and Lines, Measuring Images.

01

Algebraic Description of 2D Lines and their Homogeneous Representation

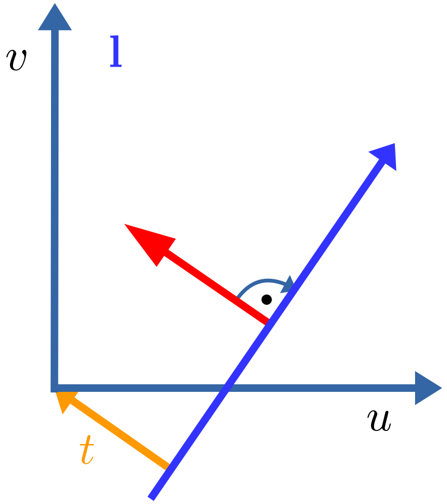
Projective Geometry of Two-Space

Algebraic Description of 2D Lines and their Homogeneous Representation

Homogeneous Equation of Lines

Points on line $\mathbf{l}$: $l_0 u + l_1 v + l_2 = 0$

Normal $\begin{pmatrix} l_0 \\ l_1 \end{pmatrix}$ Distance $t = \dfrac{-l_2}{\sqrt{l_0^2 + l_1^2}}$



Projective Geometry of Two-Space

Algebraic Description of 2D Lines and their Homogeneous Representation

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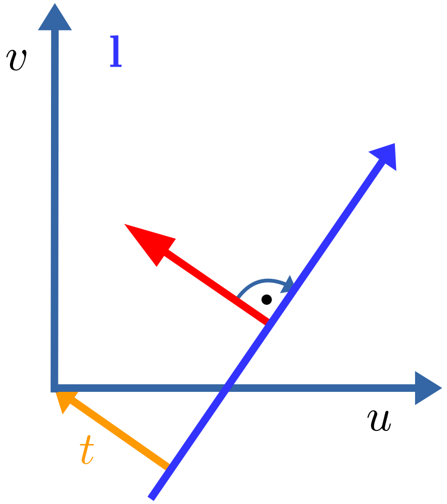

Homogeneous Coordinates of Lines

$$\mathbf{l} \cong \begin{pmatrix} l_0 \\ l_1 \\ l_2 \end{pmatrix} \in \mathbb{P}^2$$

where $\cong$ denotes “equality up scale”

- $\mathbb{R}^3$ without zero-vector
- up to **positive** scalar multiplication

Projective Geometry of Two-Space

Algebraic Description of 2D Lines and their Homogeneous Representation

Homogeneous Equation of Lines

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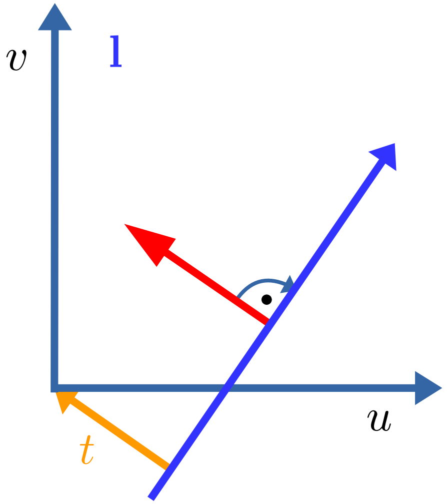

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- up to **positive** scalar multiplication

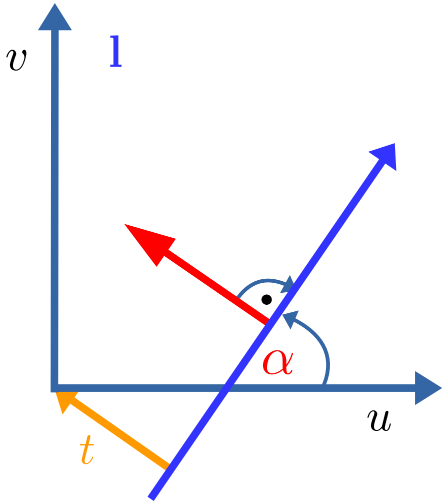
NOTE: oriented projective geometry: $\mathbf{l} \ncong -\mathbf{l}$

Projective Geometry of Two-Space

Algebraic Description of 2D Lines and their Homogeneous Representation

Line representation (summary)

$$\mathbf{l} \cong \begin{pmatrix} l_0 \\ l_1 \\ l_2 \end{pmatrix} \cong \mathrm{line}(\alpha, t) = \begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \\ -t \end{pmatrix} = \frac{1}{\sqrt{l_0^2 + l_1^2}} \begin{pmatrix} l_0 \\ l_1 \\ l_2 \end{pmatrix} \in \mathbb{P}^2$$



$\mathbf{l}$: $l_0 u + l_1 v + l_2 = 0$

α : angle to u-axis (ccw)

t : *oriented* distance to origin

NOTE: oriented projective geometry: $\mathbf{l} \not\cong -\mathbf{l}$

02

Homogeneous Representatoin of Points and Duality

Projective Geometry of Two-Space

Homogeneous Representatoin of Points and Duality

Homogeneous Coordinates of Points

$$\begin{pmatrix} u \\ v \end{pmatrix} \in \color{blue}{\mathbf{l}} \iff \boxed{(u,\ v,\ 1)} \begin{pmatrix} \color{red}{l_0} \\ \color{red}{l_1} \\ \color{orange}{l_2} \end{pmatrix} = 0$$

- Zero-vector does not represent a point.
- Scalar multiples represent same point.

$$\boxed{\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2 \leftrightarrow \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \cong \begin{pmatrix} \lambda u \\ \lambda v \\ \lambda \end{pmatrix} \in \mathbb{P}^2}$$

Projective Geometry of Two-Space

Homogeneous Representatoin of Points and Duality

Homogeneous Coordinates of Points

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Duality

→ Looks familiar?

Representation of both

points and lines in $\mathbb{P}^2$!

03

Points at Infinity and Direction

Projective Geometry of Two-Space

Points at Infinity and Direction

What about zero in last component of points?

$$\boxed{(u,\ v,\ 0)}\begin{pmatrix} l_0 \\ l_1 \\ l_2 \end{pmatrix} = 0$$

- Does not represent a point $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$

Projective Geometry of Two-Space

Points at Infinity and Direction

What about zero in last component of points?

$$\boxed{(u,\, v,\, 0)} \begin{pmatrix} l_0 \\ l_1 \\ l_2 \end{pmatrix} = 0$$

- Does not represent a point $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathbb{R}^2$
- l_2 is multiplied by zero!

  $\Rightarrow$ incident with all parallel lines with $\boxed{\text{direction} \quad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} l_1 \\ -l_0 \end{pmatrix}}$

Projective Geometry of Two-Space

Points at Infinity and Direction

→ Parallel lines intersect in an infinite point
→ Infinite points can be understood as directions
→ All infinite points lie on a line

→ The line at infinity is $\mathbf{l}^{\infty} \cong (0,0,1)^{\top}$

“Direction is the intersection with the line at infinity”

Projective Geometry of Two-Space

Points at Infinity and Direction

What about negative values in last component of points?

$$\boxed{\mathbf{x}^\top \mathbf{l} = 0 \iff -\mathbf{x}^\top \mathbf{l} = 0}$$

- Represents **the same point** $\begin{pmatrix} u \\ v \\ \pm 1 \end{pmatrix} \in \mathbb{R}^2$

- So why negative points? (coming up...)

04

Incedence, Orthogonality and Duality: Join and Meet

Projective Geometry of Two-Space

Incidence, Orthogonality and Duality: Join and Meet

- Two points define a line (“join”)

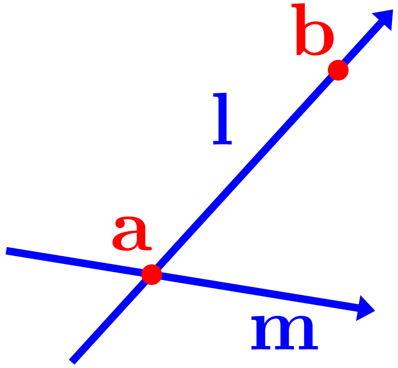
$$\mathbf{a},\ \mathbf{b} \in \mathbb{P}^2 \ \text{ are joined by } \mathbf{l} \in \mathbb{P}^2$$

$$\Longleftrightarrow \boxed{\mathbf{a}^\top \mathbf{l} = 0 \text{ and } \mathbf{b}^\top \mathbf{l} = 0}$$

- Two lines intersect in one point (“meet”)

$$\mathbf{l},\ \mathbf{m} \in \mathbb{P}^2 \ \text{ meet in } \mathbf{a} \in \mathbb{P}^2$$

$$\Longleftrightarrow \boxed{\mathbf{l}^\top \mathbf{a} = 0 \text{ and } \mathbf{m}^\top \mathbf{a} = 0}$$



Projective Geometry of Two- and Three-Space

Incedence, Orthogonality and Duality: Join and Meet

- Two points define a line (“join”)

$$\mathbf{a},\ \mathbf{b} \in \mathbb{P}^2 \ \text{ are joined by } \mathbf{l} \in \mathbb{P}^2$$

$$\iff \boxed{\mathbf{l} \cong \mathbf{a} \times \mathbf{b}}$$

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$$\mathbf{l},\ \mathbf{m} \in \mathbb{P}^2 \ \text{ meet in } \mathbf{a} \in \mathbb{P}^2$$

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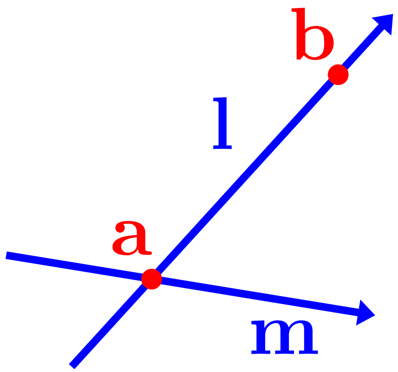

Projective Geometry of Two- and Three-Space

Incedence, Orthogonality and Duality: Join and Meet

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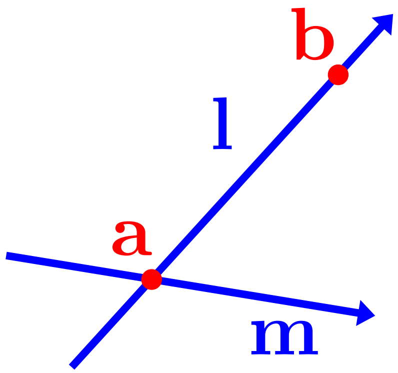

it's enough to do the math for one case and duality will give us the other...

Projective Geometry of Two- and Three-Space

Points at Infinity and Direction

- So why again negative points? $\begin{pmatrix} u \\ v \\ +1 \end{pmatrix} \ncong \begin{pmatrix} u \\ v \\ -1 \end{pmatrix}$

- Because they account for order of incedence and line direcion!

$$\mathbf{a} \cong \mathbf{l} \times \mathbf{m} \iff -\mathbf{a} \cong \mathbf{m} \times \mathbf{l}$$

Projective Geometry of Two-Space

Points at Infinity and Direction

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$$\mathbf{a} \cong \mathbf{l} \times \mathbf{m} \iff -\mathbf{a} \cong \mathbf{m} \times \mathbf{l}$$

(example for later: does a ray intersect a plane from “front” or “back”)

05

The Plücker Matrix in Two-Space

Projective Geometry of Two-Space

The Plücker Matrix in Two-Space

Suppose we have a line $\mathbf{l} \cong \mathrm{join}(\mathbf{a},\, \mathbf{b})$ through two points $\mathbf{a},\, \mathbf{b} \in \mathbb{P}^2$

$$\mathbf{l} \cong \mathbf{a} \times \mathbf{b} = \begin{pmatrix} a_1 b_2 - b_1 a_2 \\ b_0 a_2 - a_0 b_2 \\ a_0 b_1 - a_1 b_0 \end{pmatrix} = \begin{pmatrix} l_0 \\ l_1 \\ l_2 \end{pmatrix},$$

Projective Geometry of Two-Space

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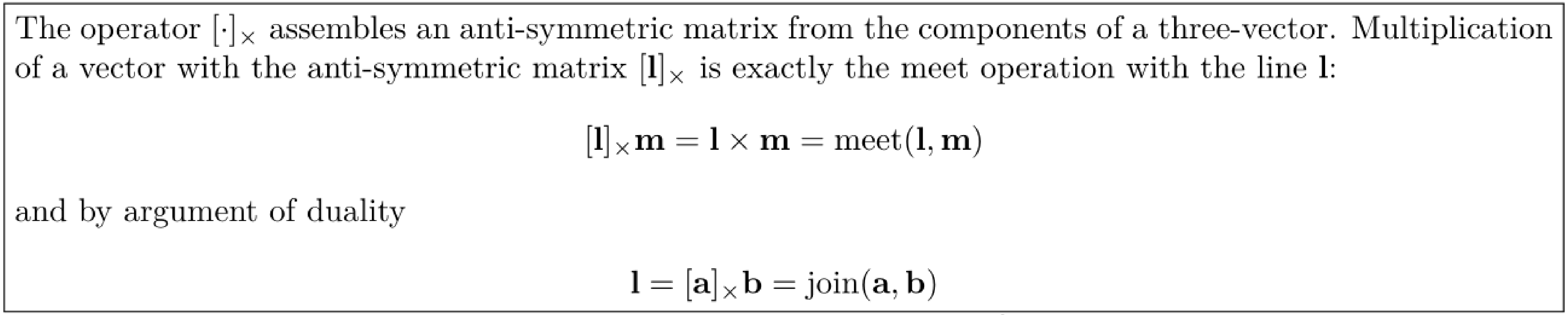
and observe that we can also compute the anti-symmetric matrix

$$= \begin{pmatrix} a_0 b_0 - b_0 a_0 & a_0 b_1 - b_0 a_1 & a_0 b_2 - b_0 a_2 \\ a_1 b_0 - b_1 a_0 & a_1 b_1 - b_1 a_1 & a_1 b_2 - b_1 a_2 \\ a_2 b_0 - b_2 a_0 & a_2 b_1 - b_2 a_1 & a_2 b_2 - b_2 a_2 \end{pmatrix} = \begin{pmatrix} 0 & l_2 & -l_1 \\ -l_2 & 0 & l_0 \\ l_1 & -l_0 & 0 \end{pmatrix}$$

Projective Geometry of Two-Space

The Plücker Matrix in Two-Space



The operator $[\cdot]_\times$ assembles an anti-symmetric matrix from the components of a three-vector. Multiplication of a vector with the anti-symmetric matrix $[\mathbf{l}]_\times$ is exactly the meet operation with the line $\mathbf{l}$:

$$[\mathbf{l}]_\times \mathbf{m} = \mathbf{l} \times \mathbf{m} = \mathrm{meet}(\mathbf{l}, \mathbf{m})$$

and by argument of duality

$$\mathbf{l} = [\mathbf{a}]_\times \mathbf{b} = \mathrm{join}(\mathbf{a}, \mathbf{b})$$

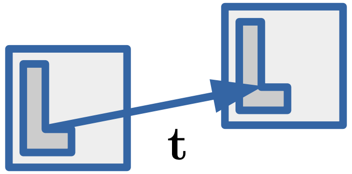
06

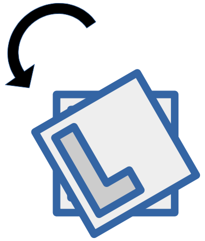
Ontology of Linear Transformations and Their Invariants

Projective Geometry of Two-Space

Ontology of Linear Transformations and Their Invariants

Homography: transformation described by a regular matrix





- Translation

$$\mathbf{t} \in \mathbb{R}^2$$

$$\downarrow$$

$$\begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \mathbf{t} \\ & & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$$

- Rotation

$$\mathbf{R}^\top \mathbf{R} = \mathrm{id} \in \mathbb{R}^{2\times 2}, \quad \det(\mathbf{R}) = 1$$

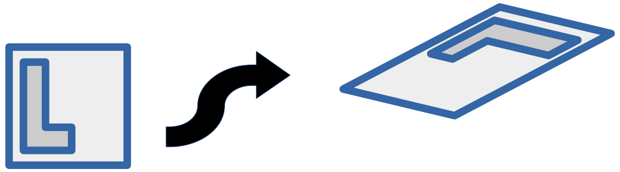
$$\downarrow$$

$$\begin{pmatrix} u' \\ v' \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R} & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$$

Projective Geometry of Two-Space

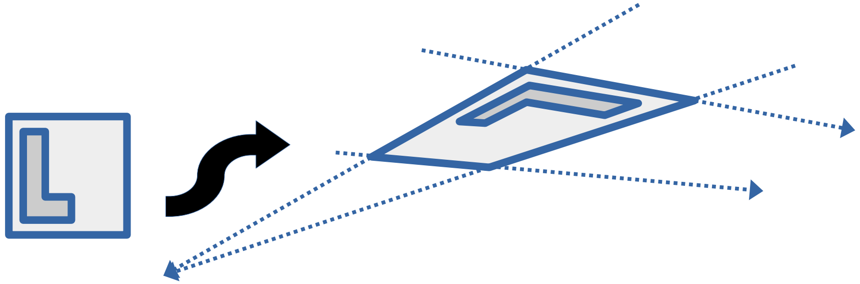
Ontology of Linear Transformations and Their Invariants

Homography: transformation described by a regular matrix



- Affine

$$\begin{pmatrix}u'\\ v'\\ 1\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{t}\\ \boxed{=0} & 1\end{pmatrix}\mathbf{x}$$



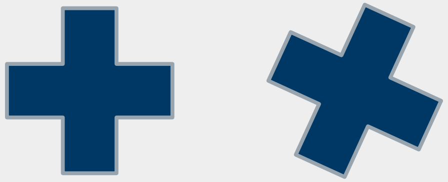
- General Projective Transformation

$$\begin{pmatrix}u'\\ v'\\ 1\end{pmatrix} \cong \mathbf{H}\mathbf{x}, \quad \det(\mathbf{H}) \neq 0$$

Projective Geometry of Two-Space

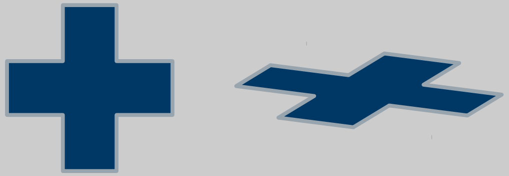
Ontology of Linear Transformations and Their Invariants

- Rigid (Translation and Rotation)
  … length, area

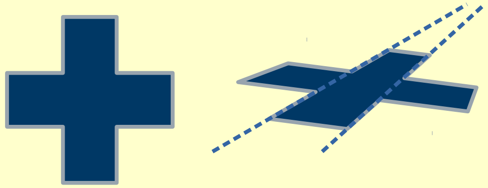


- Similarity (+Scale)
  … angles

- Affinity (+Shear)
  … parallelism



- Projective (+Perspectivity)
  … cross-ratio

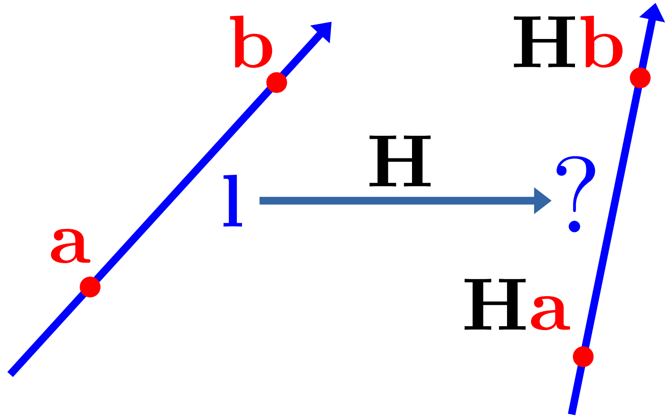


Projective Geometry of Two-Space

Ontology of Linear Transformations and Their Invariants

- Let’s consider two points joined by a line

$$\mathbf{a},\ \mathbf{b} \in \mathbb{P}^2 \ \text{ are joined by } \mathbf{l} \in \mathbb{P}^2$$
$$\iff \ \mathbf{l} \cong \mathbf{a} \times \mathbf{b}$$



- What happens to the line when we transform the points with $\mathbf{H}$

$$(\mathbf{Ha}) \times (\mathbf{Hb}) = \det(\mathbf{H})\mathbf{H}^{-\top}(\mathbf{a} \times \mathbf{b}) \cong \boxed{\mathbf{H}^{-\top}\mathbf{l}.}$$

- True also for higher dimensions (but beyond the scope)

NOTE: orientation is preserved iff $\det(\mathbf{H}) > 0$

Projective Geometry of Two-Space

Ontology of Linear Transformations and Their Invariants

- Example: Point homography for a rotation

$$\mathbf{H} \cong \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & t_x \\ -\sin(\alpha) & \cos(\alpha) & t_y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R} & \boldsymbol{t} \\ 0\,0 & 1 \end{pmatrix},$$

Transforms lines as:

$$\mathbf{H}^{-\top} \cong \begin{pmatrix} \mathbf{R} & \begin{matrix} 0 \\ 0 \end{matrix} \\ (-\mathbf{R}^\top \boldsymbol{t})^\top & 1 \end{pmatrix}.$$

Projective Geometry of Two-Space

Ontology of Linear Transformations and Their Invariants

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NOTES:

- $\mathbf{R}^\top = \mathbf{R}^{-1}$
- Normal transforms like an infinite point!
- (read: top-left 2x2 matrix is still just $\mathbf{R}$)

Practical Training Part I

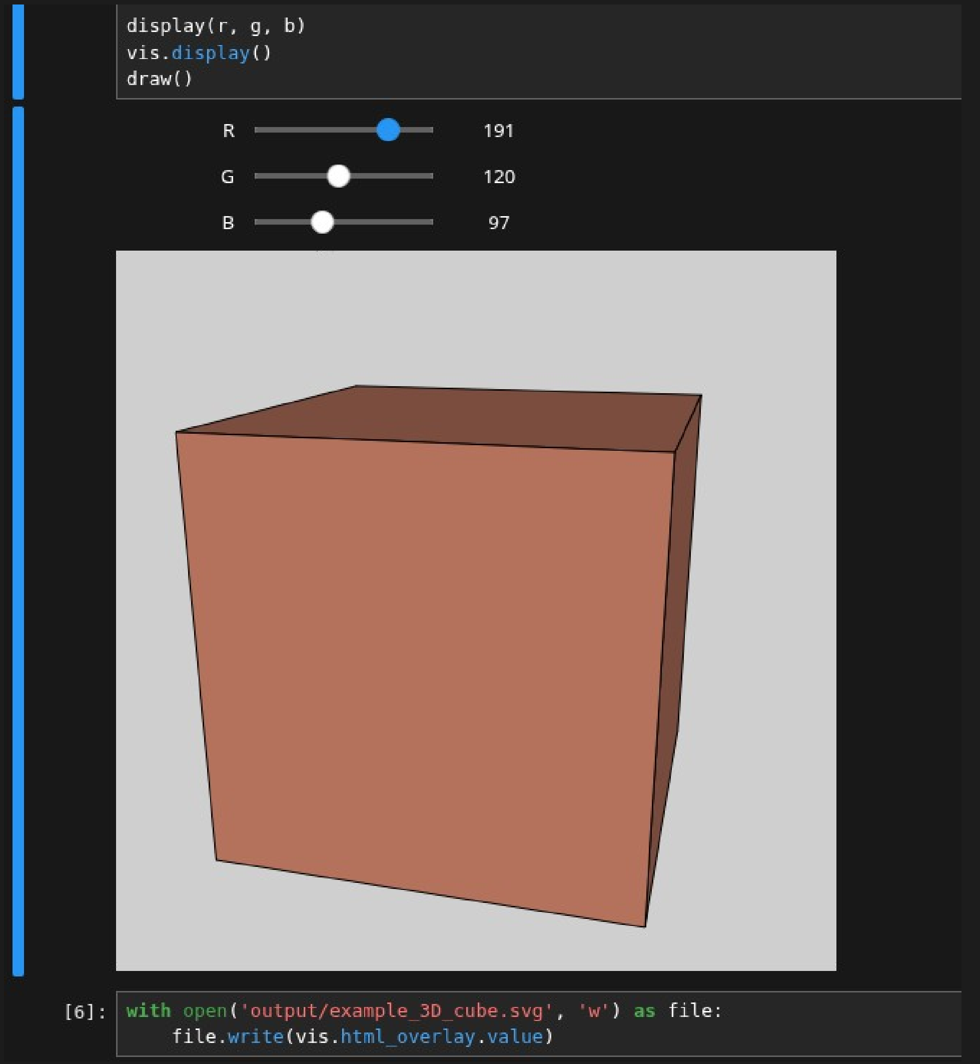
Setup Jupyter, Computing with 2D Points and Lines, Measuring Images.

01 Install Python + JupyterLab
https://www.anaconda.com/download
- OR -
pip install jupyterlab

02 Follow instructions:
https://aaichert.de/seminar

03 Open and run
01_getting_started.ipynb

04 Once that works, follow the other notebooks in order!



```
display(r, g, b)
vis.display()
draw()
```

R 191
G 120
B 97

```
[6]: with open('output/example_3D_cube.svg', 'w') as file:
         file.write(vis.html_overlay.value)
```

Projective Geometry of Three-Space

01 Duality of Points and Planes in Three-Space

02 Lines in Three-Space and the Plücker Matrix

03 Dual Plücker Coordinates: Working with Planes

04 Geometric Interpretation of the Plücker Matrix

05 Summary

Practical: Transformations, Rotating Planes and Orthogonal Projection

01

Duality of Points and Planes in Three-Space

Projective Geometry of Three-Space

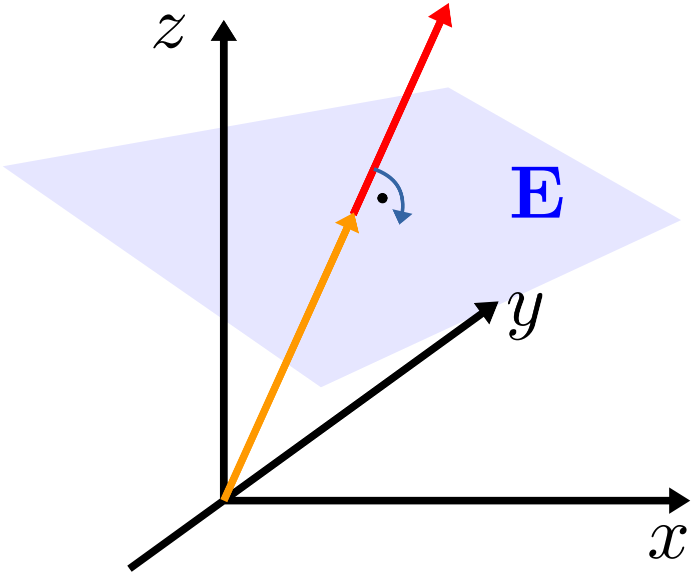
Duality of Points and Planes in Three-Space

Points on line $\mathbf{l}$: $$l_0 u + l_1 v + l_2 = 0$$

Points on plane $\mathbf{E}$: $$e_0 x + e_1 y + e_2 z + e_3 = 0$$

Plane equation and Hessian Normal Form

$$\mathbf{E} \cong \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \cong \begin{pmatrix} | \\ \mathbf{n} \\ | \\ -t \end{pmatrix} = \frac{1}{\sqrt{a^2+b^2+c^2}} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$



Projective Geometry of Three-Space

Duality of Points and Planes in Three-Space

We now have identical representation of:

$$\text{Points } \mathbf{A} \cong \begin{pmatrix} A_0 \\ A_1 \\ A_2 \\ A_3 \end{pmatrix} \text{ and } \mathbf{B} \text{ accordingly}$$

$$\text{Plane } \mathbf{P} \cong \begin{pmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{pmatrix} \text{ and } \mathbf{Q} \text{ accordingly}$$

Projective Geometry of Three-Space

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Analogously to two space, we have:

infinte points

$$(x, y, z, 0)^\top$$

that all lie on the plane at infinity

$$\pi^\infty \cong (0, 0, 0, 1)^\top$$

Projective Geometry of Three-Space

Duality of Points and Planes in Three-Space

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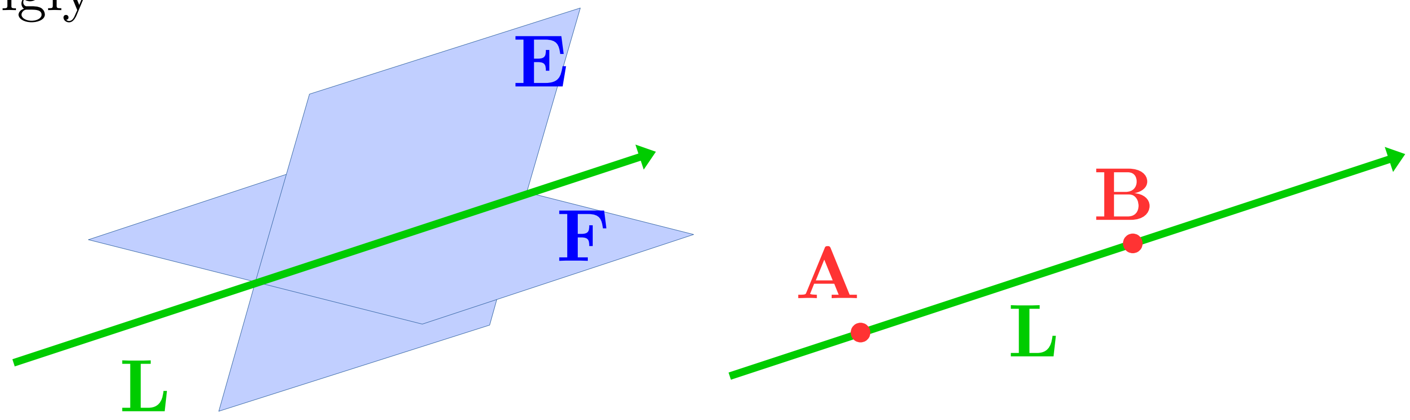
Analogously to two space, we have:

infinte points

$$(x, y, z, 0)^\top$$

that all lie on the plane at infinity

$$\pi^\infty \cong (0, 0, 0, 1)^\top$$



Now what about lines in three-space?

Projective Geometry of Three-Space

Duality of Points and Planes in Three-Space

Idea: Use Plücker Matrix to Represent 3D Lines

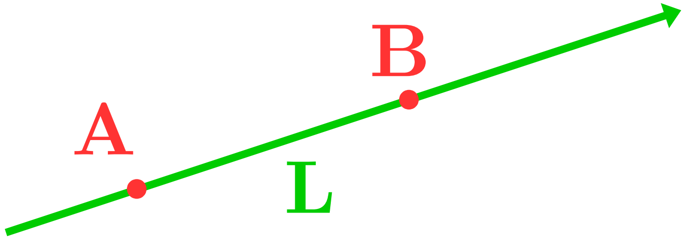
$$[\mathbf{L}]_\times = \mathbf{B}\mathbf{A}^\top - \mathbf{A}\mathbf{B}^\top \cong \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ -L_{01} & 0 & L_{12} & L_{13} \\ -L_{02} & -L_{12} & 0 & L_{23} \\ -L_{03} & -L_{13} & -L_{23} & 0 \end{pmatrix}$$

The six distinct values are called Plücker coordinates

$$\mathbf{L} \cong \begin{pmatrix} L_{01} & L_{02} & L_{03} & L_{12} & L_{13} & L_{23} \end{pmatrix}^\top$$

with $L_{01} L_{23} - L_{02} L_{13} + L_{12} L_{03} = 0.$

(a Grassmann-Plücker relation)



Projective Geometry of Three-Space

Duality of Points and Planes in Three-Space

Why is the Grassmann-Plücker relation? Simply multiply out…

$$[\mathbf{L}]_\times = \mathbf{B}\mathbf{A}^\top - \mathbf{A}\mathbf{B}^\top$$

$$\cong \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ -L_{01} & 0 & L_{12} & L_{13} \\ -L_{02} & -L_{12} & 0 & L_{23} \\ -L_{03} & -L_{13} & -L_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & (a_0b_1 - a_1b_0) & (a_0b_2 - a_2b_0) & (a_0b_3 - a_3b_0) \\ -(a_0b_1 - a_1b_0) & 0 & (a_1b_2 - a_2b_1) & (a_1b_3 - a_3b_1) \\ -(a_0b_2 - a_2b_0) & -(a_1b_2 - a_2b_1) & 0 & (a_2b_3 - a_3b_2) \\ -(a_0b_3 - a_3b_0) & -(a_1b_3 - a_3b_1) & -(a_2b_3 - a_3b_2) & 0 \end{pmatrix}$$

GPR: $(a_0b_1 - a_1b_0)(a_2b_3 - a_3b_2) - (a_0b_2 - a_2b_0)(a_1b_3 - a_3b_1) + (a_0b_3 - a_3b_0)(a_1b_2 - a_2b_1)$

$$\begin{aligned} &= (\boxed{a_0b_1a_2b_3} - a_0b_1a_3b_2 - a_1b_0a_2b_3 + a_1b_0a_3b_2) \\ &\quad - (a_0b_2a_1b_3 - a_0b_2a_3b_1 - a_2b_0a_1b_3 + a_2b_0a_3b_1) \\ &\quad + (a_0b_3a_1b_2 \boxed{- a_0b_3a_2b_1} - a_3b_0a_1b_2 + a_3b_0a_2b_1) \\ &= 0 \end{aligned}$$

02

Lines in Three-Space and the Plücker Matrix

Projective Geometry of Three-Space

Lines in Three-Space and the Plücker Matrix

Uniqueness

- Two arbitrary distinct points can be written as linear combination

$$\mathbf{A}' = \lambda_A \mathbf{A} + \lambda_B \mathbf{B}$$

$$\text{and } \mathbf{B}' = \mu_A \mathbf{A} + \mu_B \mathbf{B}.$$

- We have:

$$[\mathbf{L}']_\times = \mathbf{B}'\mathbf{A}'^\top - \mathbf{A}'\mathbf{B}'^\top = \underbrace{(\lambda_A \mu_B - \lambda_B \mu_A)}_{\neq 0 \text{ for } \mathbf{A} \not\cong \mathbf{B}} [\mathbf{L}]_\times$$

(which is identical up to sign/orientation)

Projective Geometry of Three-Space

Lines in Three-Space and the Plücker Matrix

Let's multiply a plane and see what the result is:

$$\mathbf{X} = [\mathbf{L}]_\times \mathbf{E} = \mathbf{B}\underbrace{\mathbf{A}^\top \mathbf{E}}_{\alpha} - \mathbf{A}\underbrace{\mathbf{B}^\top \mathbf{E}}_{\beta} = \mathbf{B}\alpha - \mathbf{A}\beta.$$

Where is this point?

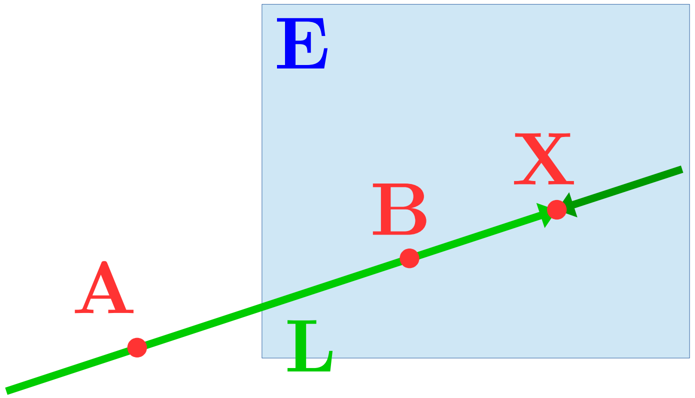
$\rightarrow$ linear combination!

$\rightarrow$ $\mathbf{X}$ is on the line $\mathbf{L}$

and $\mathbf{E}^\top \mathbf{X} = \mathbf{E}^\top [\mathbf{L}]_\times \mathbf{E} = \beta\alpha - \alpha\beta = 0.$

$\rightarrow$ $\mathbf{X}$ is on the plane $\mathbf{P}$

… it must be the intersectoin.



04

Dual Plücker Coordinates: Working with Planes

Projective Geometry of Three-Space

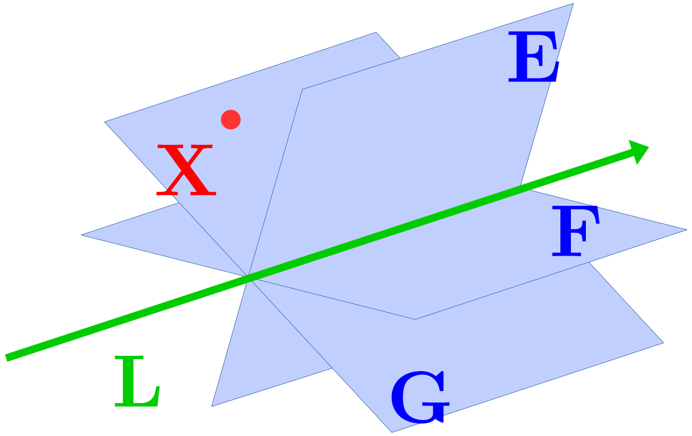
Dual Plücker Coordinates: Working with Planes

The meet of line and plane is

$$\mathbf{X} = \mathrm{meet}(\mathbf{L}, \mathbf{E}) = [\mathbf{L}]_\times \mathbf{E}.$$

But what about the duality of points and planes?

$$[\tilde{\mathbf{L}}]_\times = \mathbf{F}\mathbf{E}^\top - \mathbf{E}\mathbf{F}^\top \in \mathbb{R}^{4\times 4},$$



Projective Geometry of Three-Space

Dual Plücker Coordinates: Working with Planes

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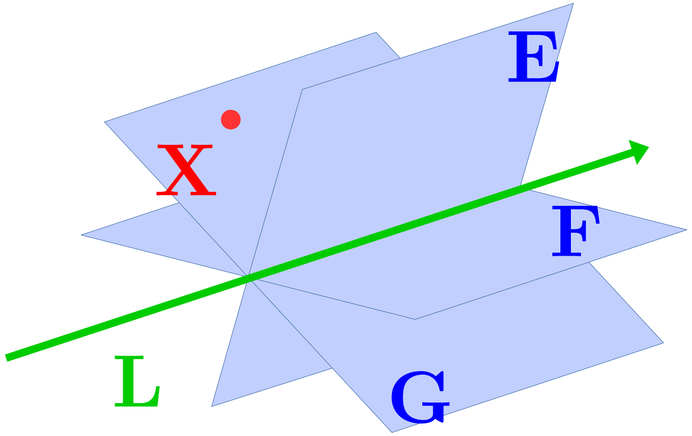
$$[\tilde{\mathbf{L}}]_\times = \mathbf{F}\mathbf{E}^\top - \mathbf{E}\mathbf{F}^\top \in \mathbb{R}^{4\times 4},$$

By the same argumenation as on the last slide:

$$\mathbf{G} = [\tilde{\mathbf{L}}]_\times \mathbf{X} = \mathrm{join}(\mathbf{L}, \mathbf{X}).$$



But how are $[\mathbf{L}]_\times$ and $[\tilde{\mathbf{L}}]_\times$ related?

(beyond the scope of this seminar :))

Projective Geometry of Three-Space

Dual Plücker Coordinates: Working with Planes

The vector $\mathbf{X} = [\mathbf{L}]_\times \mathbf{E}$ for any plane $\mathbf{E}$ is either zero or represents a point on the line.

We have: $\forall \mathbf{X} = [\mathbf{L}]_\times \mathbf{E} \in \mathbb{P}^3 : \quad [\tilde{\mathbf{L}}]_\times \mathbf{X} = 0.$

i.e.

$$\left([\tilde{\mathbf{L}}]_\times [\mathbf{L}]_\times\right)^\top = [\tilde{\mathbf{L}}]_\times [\mathbf{L}]_\times = \mathbf{0} \in \mathbb{R}^{4\times 4}.$$

The following product fulfills this property (multiply out, you get the GPR on the diagonal):

$$\begin{pmatrix} 0 & L_{23} & -L_{13} & L_{12} \\ -L_{23} & 0 & L_{03} & -L_{02} \\ L_{13} & -L_{03} & 0 & L_{01} \\ -L_{12} & L_{02} & -L_{01} & 0 \end{pmatrix} \begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ -L_{01} & 0 & L_{12} & L_{13} \\ -L_{02} & -L_{12} & 0 & L_{23} \\ -L_{03} & -L_{13} & -L_{23} & 0 \end{pmatrix} = \mathbf{0}^{4\times 4}.$$

Projective Geometry of Three-Space

Dual Plücker Coordinates: Working with Planes

The Plücker coordinates

$$\mathbf{L} = \left(L_{01}, L_{02}, L_{03}, L_{12}, L_{31}, L_{23} \right)^\top ,$$

Have a dual reresentation:

$$\tilde{\mathbf{L}} = \left(L_{23}, -L_{13}, L_{12}, L_{03}, -L_{02}, L_{01} \right)^\top .$$

There is much more to be said!

J. F. Blinn "A homogeneous formulation for lines in 3 space" SIGGRAPH Comput. Graph., vol. 11, no. 2, pp. 237–241, Jul. 1977.

Projective Geometry of Three-Space

Dual Plücker Coordinates: Working with Planes

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Have a dual reresentation:

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For “line meet plane” we use the primal:

$$\mathbf{X} = \mathrm{meet}(\mathbf{L}, \mathbf{E}) = [\mathbf{L}]_\times \mathbf{E}.$$

For “line join point” we use the dual:

$$\mathbf{G} = \mathrm{join}(\mathbf{L}, \mathbf{X}) = [\tilde{\mathbf{L}}]_\times \mathbf{X}.$$

There is much more to be said!

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04

Geometric Interpretation of the Plücker Matrix

Projective Geometry of Three-Space

Geometric Interpretation of the Plücker Matrix

Recall that infinite points are directions. Idea: Intersect line with plane at infinity:

$$\mathbf{D} = \mathrm{meet}(\mathbf{L},\, \pi^\infty) = [\mathbf{L}]_\times \cdot \pi^\infty = \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix}.$$

Projective Geometry of Three-Space

Geometric Interpretation of the Plücker Matrix

Recall that infinite points are directions. Idea: Intersect line with plane at infinity:

$$\mathbf{D} = \operatorname{meet}(\mathbf{L}, \pi^\infty) = [\mathbf{L}]_\times \cdot \pi^\infty = \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix}.$$

Dually, we can compute a plane which contains the line and the origin:

$$\mathbf{M} = \operatorname{join}\left(\mathbf{L}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right) = [\tilde{\mathbf{L}}]_\times \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathbf{m} \\ 0 \end{pmatrix}.$$

Projective Geometry of Three-Space

Geometric Interpretation of the Plücker Matrix

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We get the line direction $\mathbf{d} = \begin{pmatrix} L_{03} & L_{13} & L_{23} \end{pmatrix}^{\top}$

And moment $\mathbf{m} = \begin{pmatrix} L_{01} & -L_{02} & L_{12} \end{pmatrix}^{\top}$.

Projective Geometry of Three-Space

Geometric Interpretation of the Plücker Matrix

Recall that infinite points are directions. Idea: Intersect line with plane at infinity:

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Dually, we can compute a plane which contains the line and the origin:

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We get the line direction $\mathbf{d} = \begin{pmatrix} L_{03} & L_{13} & L_{23} \end{pmatrix}^\top$

And moment $\mathbf{m} = \begin{pmatrix} L_{01} & -L_{02} & L_{12} \end{pmatrix}^\top$.

NOTE: the Grassmann-Plücker relation states that direction and moment are orthogonal: $\mathbf{d}^\top \mathbf{m} = 0$

Projective Geometry of Three-Space

Geometric Interpretation of the Plücker Matrix

We can now express the primal and dual Plücker Matrices

$$[\mathbf{L}]_\times = \begin{pmatrix} [\mathbf{m}]_\times & \mathbf{d} \\ -\mathbf{d}^\top & 1 \end{pmatrix}, \quad [\tilde{\mathbf{L}}]_\times = \begin{pmatrix} [\mathbf{d}]_\times & \mathbf{m} \\ -\mathbf{m}^\top & 1 \end{pmatrix}.$$

Projective Geometry of Three-Space

Geometric Interpretation of the Plücker Matrix

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Connection to the way Plücker coordinates are often taught in CG classes*:

Let $\mathbf{A} = (\mathbf{a}^\top, 1)^\top = (A_1, A_2, A_3, 1)^\top$ and $\mathbf{B}$ analogously denote finite points on $\mathbf{L}$

Then: $\mathbf{d} = \mathbf{b} - \mathbf{a}$ and $\mathbf{m} = \mathbf{a} \times \mathbf{b}\,.$

*** J. F. Blinn “A homogeneous formulation for lines in 3 space” SIGGRAPH Comput. Graph., vol. 11, no. 2, pp. 237–241, Jul. 1977.**

Projective Geometry of Three-Space

Geometric Interpretation of the Plücker Matrix

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Let $\mathbf{A} = (\mathbf{a}^\top, 1)^\top = (A_1,\, A_2,\, A_3, 1)^\top$ and $\mathbf{B}$ analogously denote finite points on $\mathbf{L}$

Then: $\mathbf{d} = \mathbf{b} - \mathbf{a}$ and $\mathbf{m} = \mathbf{a} \times \mathbf{b}$.

Careful: Some libraries define Plücker coordinates as $(\mathbf{m}, \mathbf{d})^\top \neq \mathbf{L}$.

This is a different convention from the algebraic ordering!

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Projective Geometry of Three-Space

Geometric Interpretation of the Plücker Matrix

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Connection to the way Plücker coordinates are often taught in CG classes*:

Let $\mathbf{A} = (\mathbf{a}^\top, 1)^\top = (A_1, A_2, A_3, 1)^\top$ and $\mathbf{B}$ analogously denote finite points on $\mathbf{L}$

Then: $\boxed{\mathbf{d} = \mathbf{b} - \mathbf{a}}$ and $\mathbf{m} = \mathbf{a} \times \mathbf{b}\,.$

Careful: Some libraries define Plücker coordinates as $(\mathbf{m}, \mathbf{d})^\top \neq \mathbf{L}$.

This is a different convention from the algebraic ordering!

Also note: $[\mathbf{L}]_\times = \mathbf{B}\mathbf{A}^\top - \mathbf{A}\mathbf{B}^\top = \boxed{-(\mathbf{A}\mathbf{B}^\top - \mathbf{B}\mathbf{A}^\top)}.$

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Projective Geometry of Three-Space

Geometric Interpretation of the Plücker Matrix

We now change perspectives from point to plane.

Consider again the infinite point on the line

$$\mathbf{D} = \operatorname{meet}(\mathbf{L}, \pi^{\infty}) = [\mathbf{L}]_{\times} \cdot \pi^{\infty} = \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix}.$$

If we think of it as a plane, where is that plane?

Projective Geometry of Three-Space

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If we think of it as a plane, where is that plane?

- The plane $\mathbf{D}$ contains the origin
- It is orthogonal to the line

Projective Geometry of Three-Space

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If we think of it as a plane, where is that plane?

- The plane $\mathbf{D}$ contains the origin
- It is orthogonal to the line

By intersection, we get the closest point to the origin

$$\mathbf{T} = [\mathbf{L}]_{\times} [\mathbf{L}]_{\times} \cdot \pi^{\infty} = \begin{pmatrix} \mathbf{d} \times \mathbf{m} \\ \|\mathbf{d}\|^2 \end{pmatrix}.$$

Projective Geometry of Three-Space

Geometric Interpretation of the Plücker Matrix

We now change perspectives from point to plane.

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- The plane $\mathbf{D}$ contains the origin
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By intersection, we get the closest point to the origin

$$\mathbf{T} = [\mathbf{L}]_\times [\mathbf{L}]_\times \cdot \pi^\infty = \begin{pmatrix} \mathbf{d} \times \mathbf{m} \\ \|\mathbf{d}\|^2 \end{pmatrix}.$$

The distance of the line to the origin is

$$d = \frac{\|\mathbf{m}\|}{\|\mathbf{d}\|}.$$

Projective Geometry of Three-Space

Summary

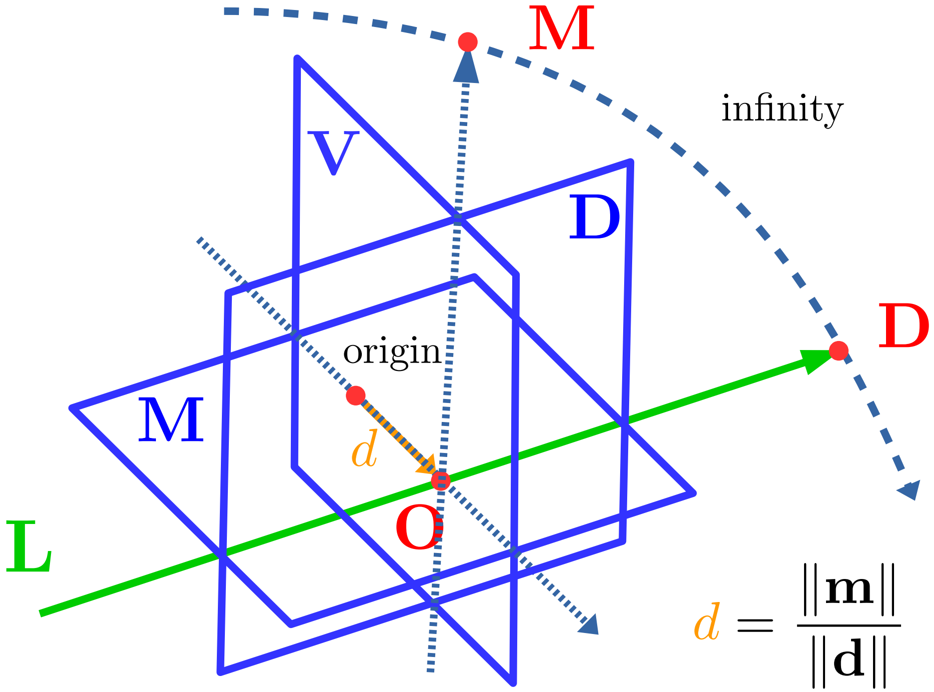
$$\mathbf{d} = (L_{03},\, L_{13},\, L_{23})^\top \qquad \mathbf{m} = (-L_{12}, L_{02}, -L_{01})^\top$$

$$[\mathbf{L}]_\times = \begin{pmatrix} \boxed{[\mathbf{m}]_\times} & \boxed{\mathbf{d}} \\ \boxed{-\mathbf{d}^\top} & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & L_{01} & L_{02} & L_{03} \\ -L_{01} & 0 & L_{12} & L_{13} \\ -L_{02} & -L_{12} & 0 & L_{23} \\ -L_{03} & -L_{13} & -L_{23} & 0 \end{pmatrix}$$

$$\mathbf{O} = [\mathbf{L}]_\times \underbrace{[\mathbf{L}]_\times \pi^\infty}_{\mathbf{D}=(\mathbf{d}^\top,0)^\top} \cong \begin{pmatrix} \boxed{[\mathbf{m}]_\times} & \boxed{\mathbf{d}} \\ \boxed{-\mathbf{d}^\top} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{d} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{m} \times \mathbf{d} \\ -\mathbf{d}^\top \mathbf{d} \end{pmatrix}$$

$$\mathbf{V} = [\tilde{\mathbf{L}}]_\times \underbrace{[\tilde{\mathbf{L}}]_\times \pi^\infty}_{\mathbf{M}=(\mathbf{m}^\top,0)^\top} \cong \begin{pmatrix} \boxed{[\mathbf{d}]_\times} & \boxed{\mathbf{m}} \\ \boxed{-\mathbf{m}^\top} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{m} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{m} \times \mathbf{d} \\ \mathbf{m}^\top \mathbf{m} \end{pmatrix}$$



$$d = \frac{\|\mathbf{m}\|}{\|\mathbf{d}\|}$$

Projective Geometry of Three-Space

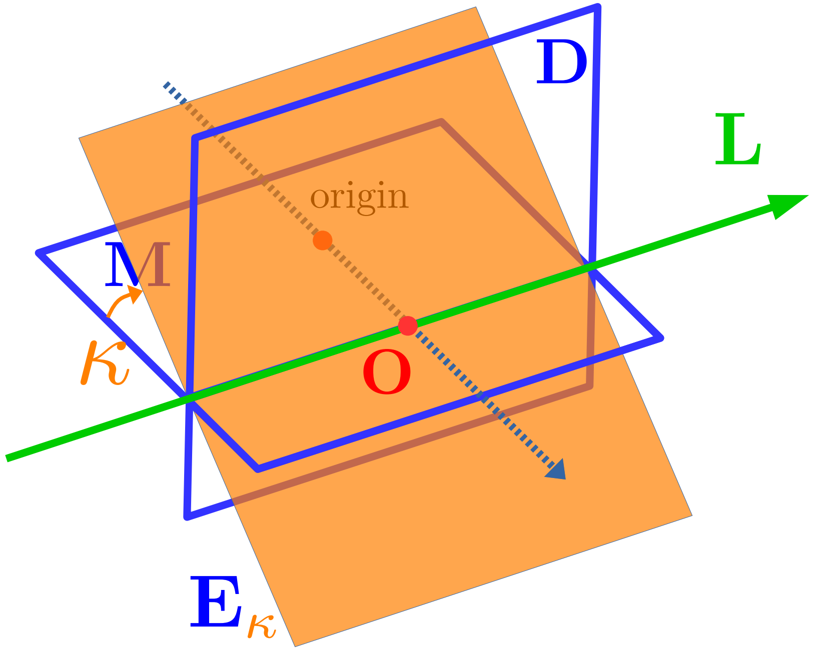
Summary

Outlook: a trick we'll use tomorrow.

If we have all planes in Hessian normal form, we can use a linear combination (i.e. matrix multiplication) to pick any plane from the pencil around $\mathbf{L}$ by an angle κ:

$$\mathbf{E}_\kappa = \begin{pmatrix} \mathbf{M} & \mathbf{D} \end{pmatrix} \begin{pmatrix} \cos(\kappa) \\ -\sin(\kappa) \end{pmatrix}$$

(which can be written directly in the Plücker coordinates)



Practical Training Part II

Geometric Interpretation of the Plücker Matrix

01 Translate and rotate a 3D line using a homography. (*)

02 Given a line, compute an orthogonal line thru the origin.

03 Implement picking a plane from the line’s pencil by angle.

04 Build a function to project points orthogonally to a plane.

05 Do the same with just one matrix.

***:**
$$\begin{aligned}\mathbf{L}' &= (\mathbf{T}\mathbf{B})(\mathbf{T}\mathbf{A})^\top - (\mathbf{T}\mathbf{A})(\mathbf{T}\mathbf{B})^\top \\ &= \mathbf{T}\mathbf{B}\mathbf{A}^\top\mathbf{T}^\top - \mathbf{T}\mathbf{A}\mathbf{B}^\top\mathbf{T}^\top \\ &= \mathbf{T}\left(\mathbf{B}\mathbf{A}^\top - \mathbf{A}\mathbf{B}^\top\right)\mathbf{T}^\top \\ &= \mathbf{T}\mathbf{L}\mathbf{T}^\top\end{aligned}$$

Single View Geometry and the Projection Matrix

01 Central Projection in Homogeneous Coordinates

02 Intrinsic Parameters and KRt Decomposition

03 Source Position and Backprojection

04 Axis Planes and Principal Plane

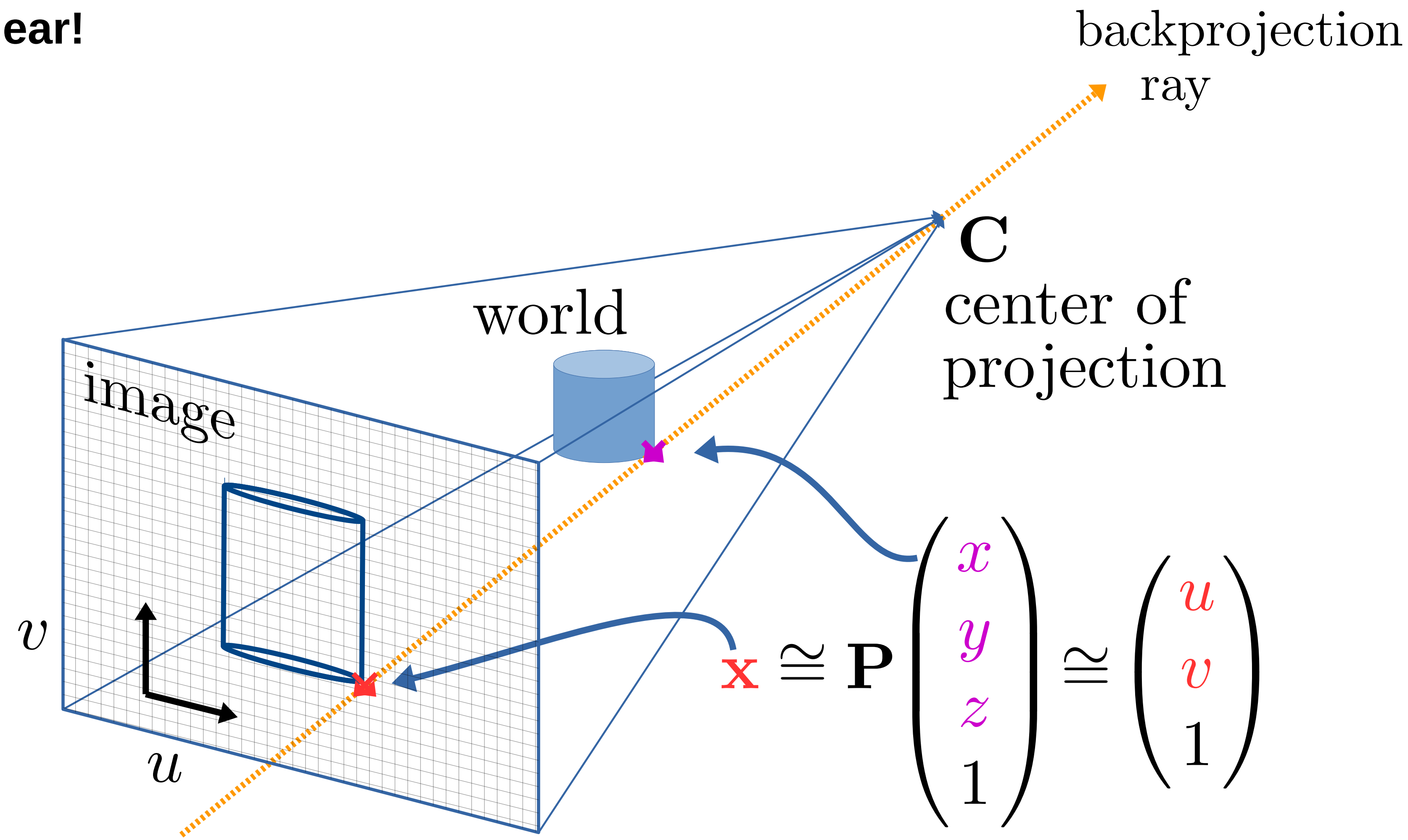
01

Central Projection in Homogeneous Cordinates

Single View Geometry and the Anatomy of the Projection Matrix

Central Projection in Homogeneous Coordinates

It’s all linear!



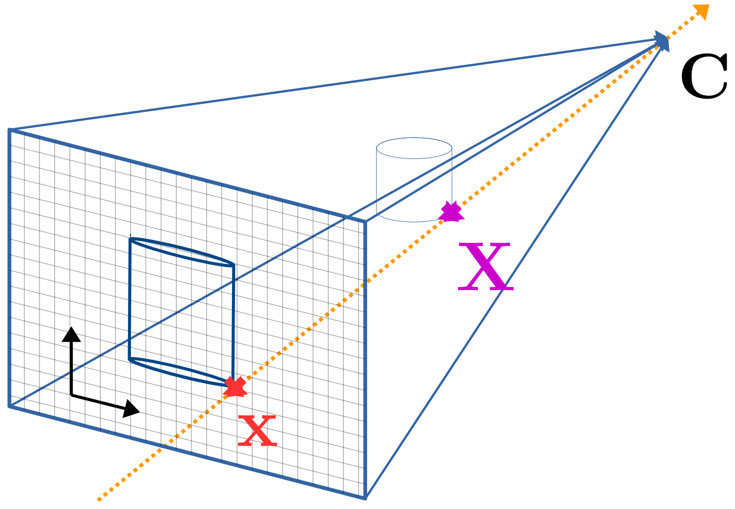
$$\mathbf{x} \cong \mathbf{P}\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \cong \begin{pmatrix} u \\ v \\ 1 \end{pmatrix}$$

Single View Geometry and the Anatomy of the Projection Matrix

Central Projection in Homogeneous Coordinates

It’s all linear!

So we should describe it by a matrix.



$$\mathbf{x} \cong \mathbf{P}\mathbf{X}$$

image point in $\mathbb{P}^2$

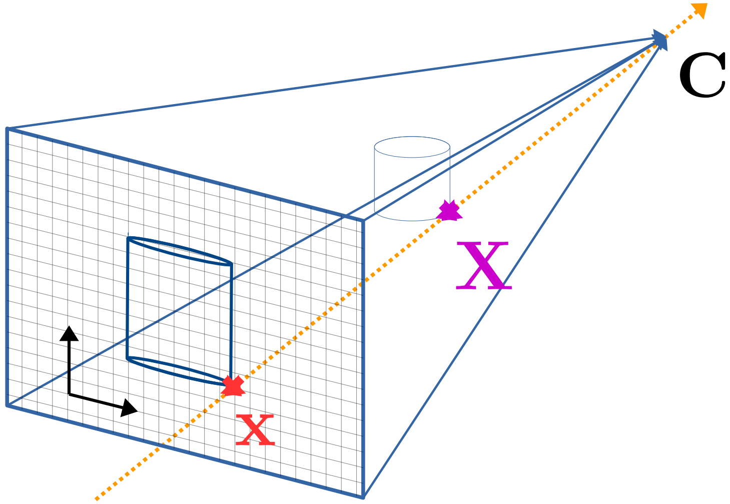
$\mathbb{R}^{3\times 4}$ up to scale

world point in $\mathbb{P}^3$

Single View Geometry and the Anatomy of the Projection Matrix

Central Projection in Homogeneous Coordinates

- Rays through camera center $\mathbf{C}$ and world point $\mathbf{X}$ intersect image plane at image point $\mathbf{x}$
- Projection described by a single 3x4 matrix up to scale
- Projects both points in “front of” and “behind” camera (we will follow up on that)



02

Intrinsic Parameters and the KRt Decomposition

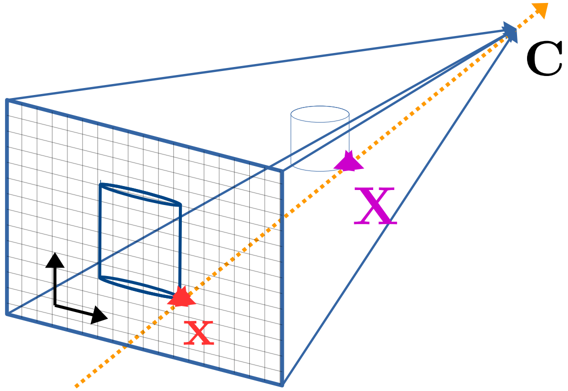
Single View Geometry and the Anatomy of the Projection Matrix

Central Projection in Homogeneous Coordinates

I skip most of the constructive approach in this seminar.

Any book on computer vision tells you about how a projectoin matrix can be (de-)composed as

$$\mathbf{P} = \mathbf{K}\begin{pmatrix}\mathbf{R} & \mathbf{t}\end{pmatrix}$$



Single View Geometry and the Anatomy of the Projection Matrix

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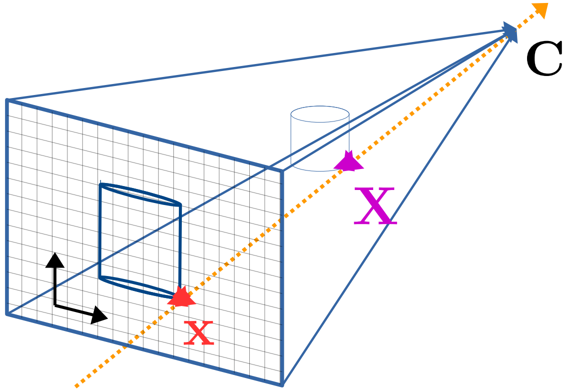
$$\mathbf{P} = \mathbf{K}\begin{pmatrix}\mathbf{R} & \mathbf{t}\end{pmatrix}$$

With intrincic parameters

$$\mathbf{K} = \begin{pmatrix}\alpha_x & s & u_0 \\ 0 & \alpha_y & v_0 \\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}f \cdot m_x & s & u_0 \\ 0 & f \cdot m_y & v_0 \\ 0 & 0 & 1\end{pmatrix}$$



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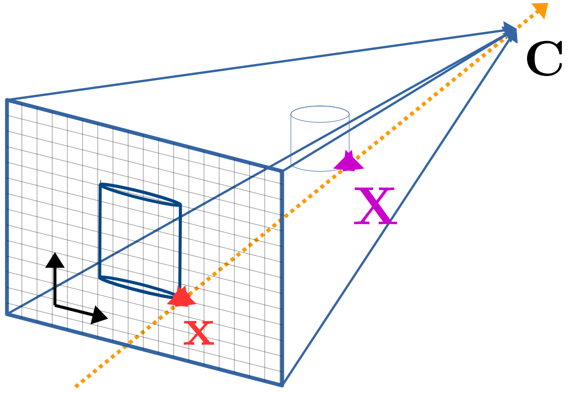
Central Projection in Homogeneous Coordinates

I skip most of the constructive approach in this seminar.

Any book on computer vision tells you about how a projectoin matrix can be (de-)composed as



$$\mathbf{P} = \mathbf{K}\begin{pmatrix}\mathbf{R} & \mathbf{t}\end{pmatrix}$$

With intrincic parameters

$$\mathbf{K} = \begin{pmatrix} \alpha_x & s & u_0 \\ 0 & \alpha_y & v_0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} f\cdot m_x & s & u_0 \\ 0 & f\cdot m_y & v_0 \\ 0 & 0 & 1 \end{pmatrix}$$

principal point (in pixels)

focal lenth (in pixels)

skew = 0

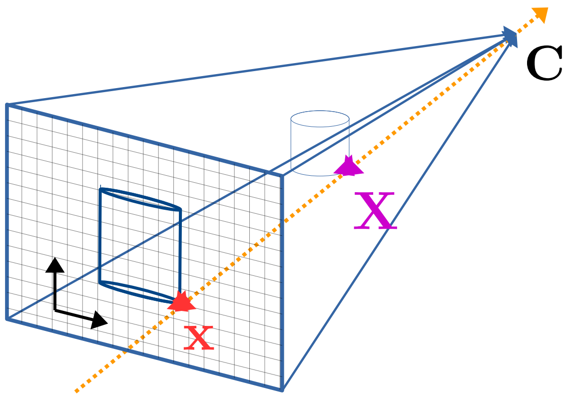
pixel spacing (usually equal)

Single View Geometry and the Anatomy of the Projection Matrix

Central Projection in Homogeneous Coordinates

If you want the decomposition: QR decomposition is usually available.
Use this matrix to get the RQ decomposition instead:

$$\mathbf{K}_{\text{raw}} = \mathbf{J}\mathbf{R}_{\text{qr}}^{\top}\mathbf{J} \quad \text{and} \quad \mathbf{R}_{\text{raw}} = \mathbf{J}\mathbf{Q}_{\text{qr}}^{\top} \;\text{ with }\; J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$



Fix ambiguity of QR: the diagonal of K must be positive.

Finally $\mathbf{t} = \mathbf{K}^{-1}\mathbf{p}_4$.

Single View Geometry and the Anatomy of the Projection Matrix

Central Projection in Homogeneous Coordinates

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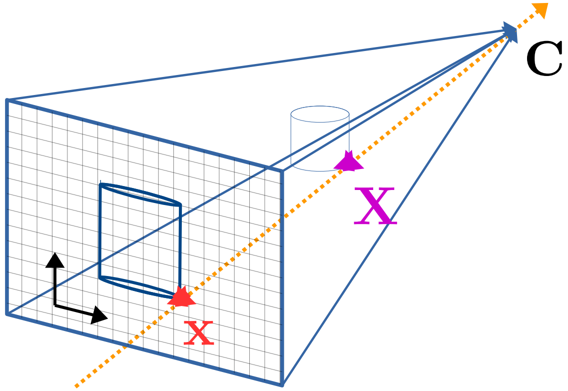


Fix ambiguity of QR: the diagonal of K must be positive.

Finally $\mathbf{t} = \mathbf{K}^{-1}\mathbf{p}_4$.

In this seminar, instead, we'll focus on the anatomy of the projection matrix.

- How to debug values I see on the command line?
- How to understand orientation and axes?
- Specifically for X-ray: where is my detector?

03

Source Position and Backprojection Rays

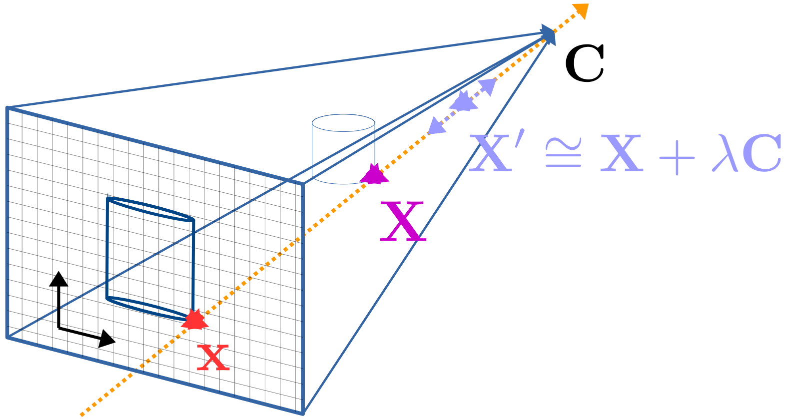
Single View Geometry and the Anatomy of the Projection Matrix

Source Position and Backprojection Rays

The center of projection is the right null-space of the projection matrix

- If $\mathbf{X}$ is a point on a projection ray, then so is $\mathbf{C}$
- All points on a ray map to the same image point
- And thus any linear combination of the two does, too:

$$\mathbf{x} \cong \mathbf{P}\left(\mathbf{X} + \lambda\mathbf{C}\right) \cong \mathbf{PX} + \underbrace{\lambda\mathbf{PC}}_{=0}$$



Single View Geometry and the Anatomy of the Projection Matrix

Source Position and Backprojection Rays

The pseudo-inverse allows us to compute a point with identical projection:

$$\mathbf{X}^+ = \mathbf{P}^+\mathbf{x}\,, \qquad \mathbf{P}\mathbf{X}^+ = \mathbf{P}\mathbf{P}^+\mathbf{x} = \mathbf{x}$$

The **backprojection ray** is defined by the camera center and another point.

$$\mathbf{R} = \mathrm{join}(\mathbf{X}^+, \mathbf{C})$$

Let $\mathbf{l}$ denote a 2D line through two image points $\mathbf{a}$ and $\mathbf{b}$.

Further, let $\mathbf{A}^+ \cong \mathbf{P}^+\mathbf{a}$ and $\mathbf{B}^+$, accordingly denote their backprojection points.

Now, the plane $\mathbf{E} \cong \mathbf{P}^\top\mathbf{l}$ is the **backprojection plane** of $\mathbf{l}$ because it contains both these points and the camera center:

$$\mathbf{E}^\top\mathbf{A}^+ = \mathbf{l}^\top \underbrace{\mathbf{P}\mathbf{P}^+}_{identity} \mathbf{a} = \mathbf{l}^\top\mathbf{a}, \qquad \mathbf{E}^\top\mathbf{C} = \mathbf{l}^\top \underbrace{\mathbf{P}\mathbf{C}}_{=0} = 0.$$

04

Axis Planes and Principal Plane

Single View Geometry and the Anatomy of the Projection Matrix

Axis Planes and Principal Plane

Rows of the matrix can be interpreted as planes

$$\mathbf{P} \cong \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{pmatrix} \qquad \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \cong \mathbf{PX} = \begin{pmatrix} - & \mathbf{p}^{1\top} & - \\ - & \mathbf{p}^{2\top} & - \\ - & \mathbf{p}^{3\top} & - \end{pmatrix} \mathbf{X}$$

Single View Geometry and the Anatomy of the Projection Matrix

Axis Planes and Principal Plane

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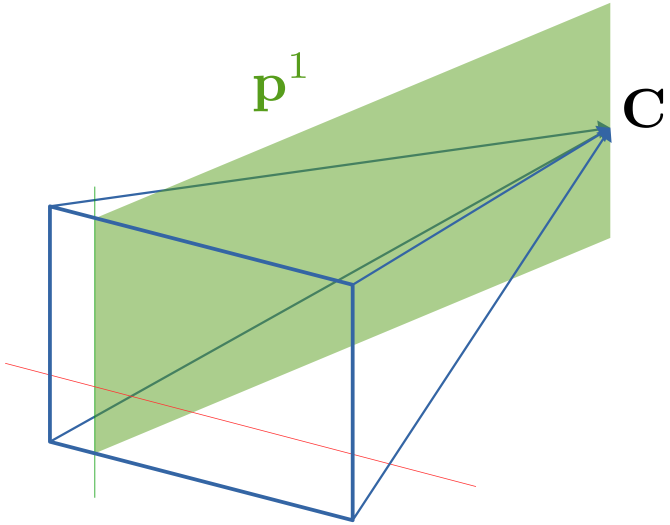
- First row: $u = \mathbf{p}^{1\top}\mathbf{X}$
- Points on $\mathbf{p}^1$ fulfill $u = 0$

$\Rightarrow$ $\mathbf{p}^1$ contains v-axis!



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Single View Geometry and the Anatomy of the Projection Matrix

Axis Planes and Principal Plane

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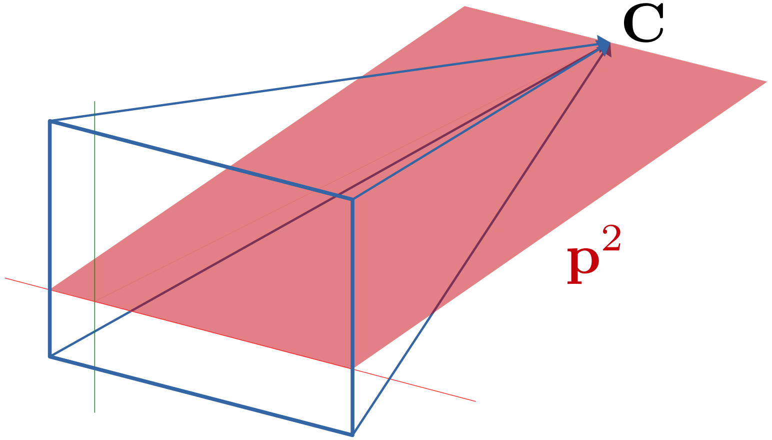
- Second row: $v = \mathbf{p}^{2\top}\mathbf{X}$
- Points on $\mathbf{p}^2$ fulfill $v = 0$

$\Rightarrow$ $\mathbf{p}^2$ contains u-axis!



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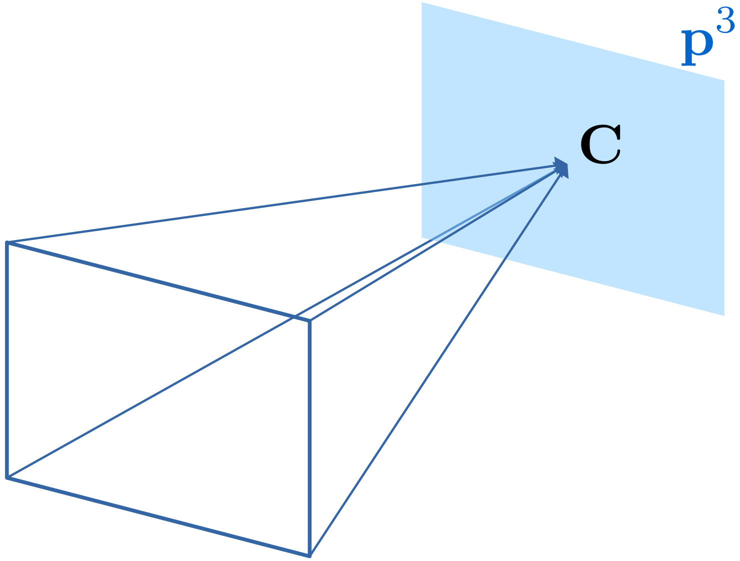
Single View Geometry and the Anatomy of the Projection Matrix

Axis Planes and Principal Plane

The principal plane

$$\mathbf{P} \cong \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{pmatrix}$$



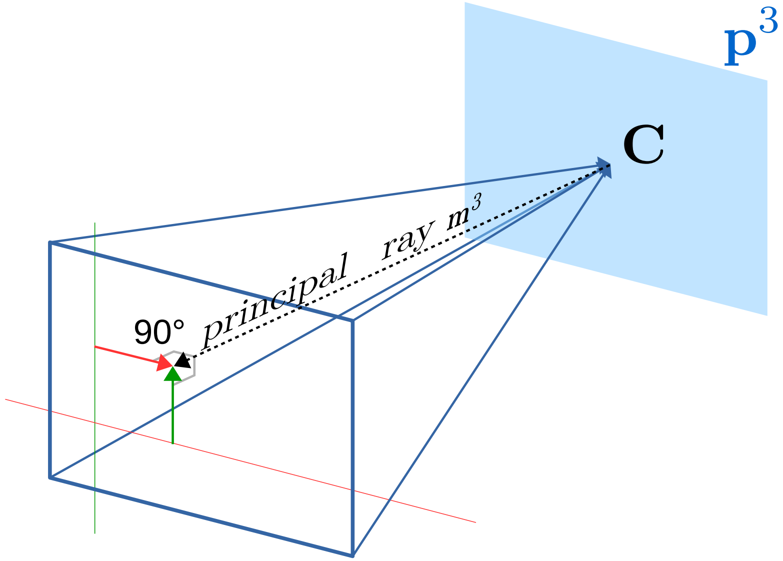
- Points on $\mathbf{p}^3$ fulfill $0 = \mathbf{p}^{3\top}\mathbf{X}$ and lie on a plane parallel to the image
- They are mapped to infinity!
- $\mathbf{C} = \mathrm{null}(\mathbf{P})$ lies on all planes.

Single View Geometry and the Anatomy of the Projection Matrix

Axis Planes and Principal Plane

The principal plane

$$\mathbf{P} \cong \begin{pmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{pmatrix}$$



The principal ray is the normal of the principal plane and is orthogonal to the image plane.

Single View Geometry and the Anatomy of the Projection Matrix

Axis Planes and Principal Plane

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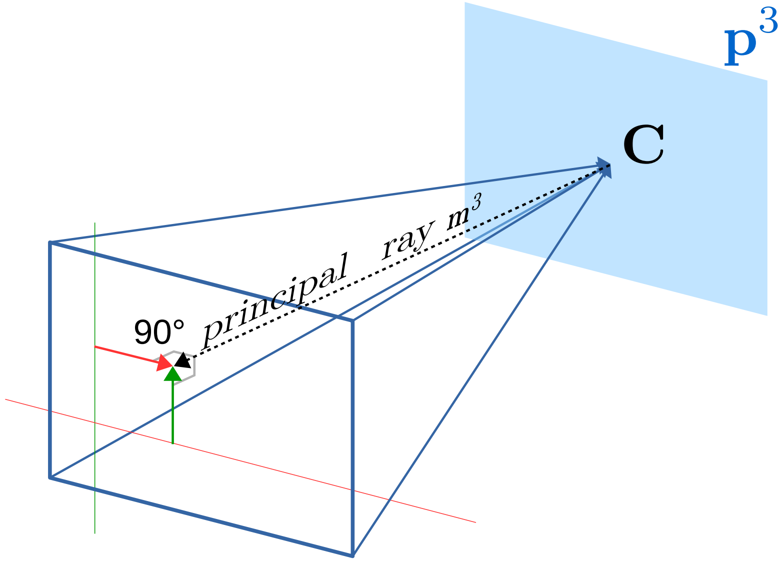
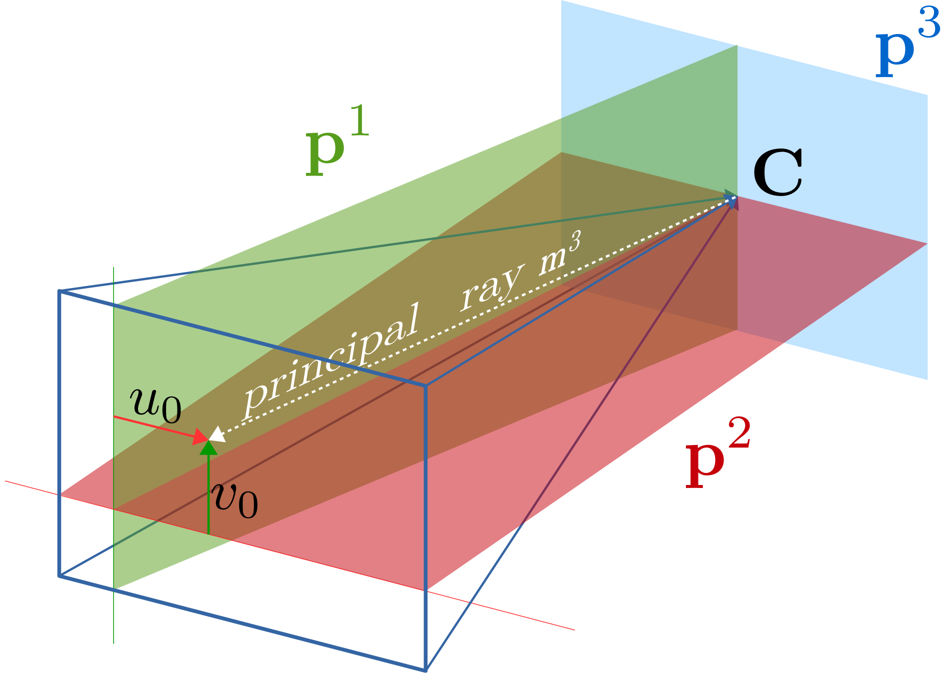


$$\begin{pmatrix} u_0 \\ v_0 \\ 1 \end{pmatrix} \cong \mathbf{M}\mathbf{m}^3$$

Single View Geometry and the Anatomy of the Projection Matrix

Axis Planes and Principal Plane

Summary



$$\mathbf{P} \cong \begin{pmatrix} \underbrace{\begin{matrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{matrix}}_{\mathbf{m}^3} & \begin{matrix} P_{14} \\ P_{24} \\ P_{34} \end{matrix} \end{pmatrix}$$

$$\begin{pmatrix} u_0 \\ v_0 \\ 1 \end{pmatrix} \cong \boxed{\mathbf{M}\mathbf{m}^3}$$

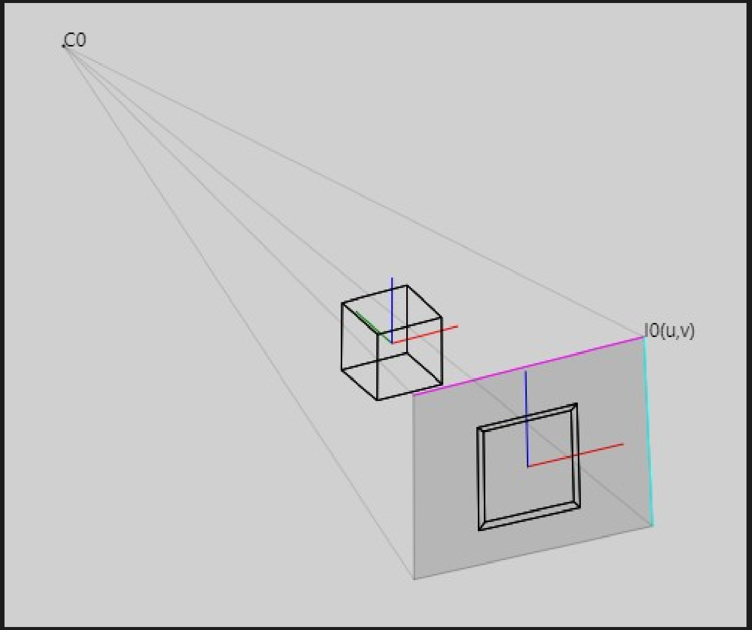
Practical Training Part III

Using Plücker lines to visualize an X-ray source-detector geometry.

01 Project a 3D line to an image with a projection matix.

02 Extract the principal plane from a projectoin matrix.

03 Compute the focal length in pixels and move the principal plane to the detector.



05 Compute the center of projection from a projectoin matrix and compute the backprojection rays for the four image corners.

06 Intersect the four backprojection rays with the detector plane.

07 Build a matrix which lifts pixels to 3D points on the detector.

03

Summary and Outlook

Summary

01 **We should always use homogeneous coordinates for linear geom..**

02 **We understand linear transformations.**

03 **We are able to write complex incedence geometry in matrix form.**

04 **We understand why lines in 3D are written as skew-sym. Matrices.**

05 **We finally understand Plucker coordinates.**

06 **When we look at a projection matrix, we have an idea what it says.**

07 **We are able to visually debug x-ray source detector geometries.**

Tomorrow: Apply your new knowledge to Pairs of X-Rays

https://github.com/aaichert/xray-epipolar-consistency/