Projective Geometry

Tutorial: Join and Meet in Two and Three Dimensions

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Pattern Recognition Lab (CS 5)

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Introduction: Coordinates of Flats

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Part I

Real Projective Space

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Lines in the projective plane

The 2D projective line

$$\mathbf{l} \cong \begin{pmatrix} l_0 \\ l_1 \\ l_2 \end{pmatrix} \cong \frac{\pm 1}{\sqrt{l_0^2 + l_1^2}} \begin{pmatrix} l_0 \\ l_1 \\ l_2 \end{pmatrix} \in \mathbb{P}^2 \sim \mathbb{S}^{2+} \times \mathbb{R}$$

where $\cong$ denotes “equality up scale”, exists in

projective two-space $\mathbb{P}^2 = \dfrac{\mathbb{R}^3 \backslash \{\mathbf{0}\}}{\mathbb{R} \backslash \{0\}}$

($\mathbb{R}^3$ without zero-vector, up to non-zero scalar multiples)

Part II

Incidence Geometry

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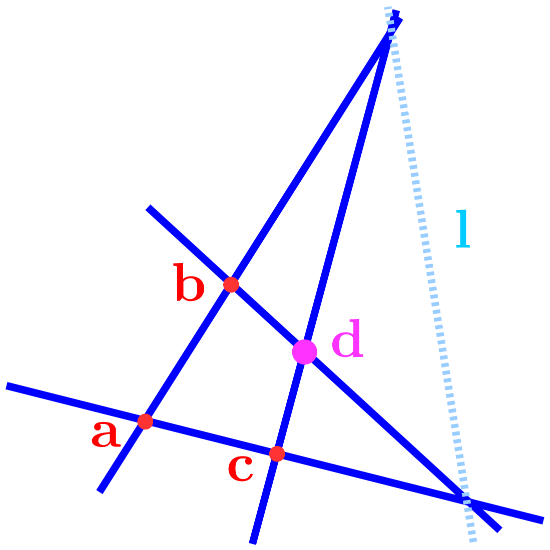
Playful example

- Given **a,b,c,d** express **l**

$$\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2 = (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) \times (\mathbf{a} \times \mathbf{c}) \times (\mathbf{b} \times \mathbf{d})$$

- Expression entirely in determinants

- Bracket algebra

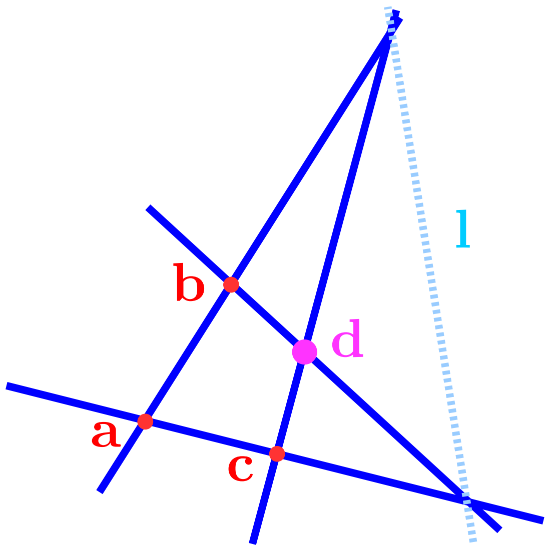


Playful example

- Given **a,b,c** express **l** by **d**

$$\mathbf{l} = \mathbf{x}_1 \times \mathbf{x}_2 = (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) \times (\mathbf{a} \times \mathbf{c}) \times (\mathbf{b} \times \mathbf{d})$$

- Expression entirely in determinants
- Bracket algebra
- **Food for CAS & automatic proofer**



Visualizing Real Projective Two-Space

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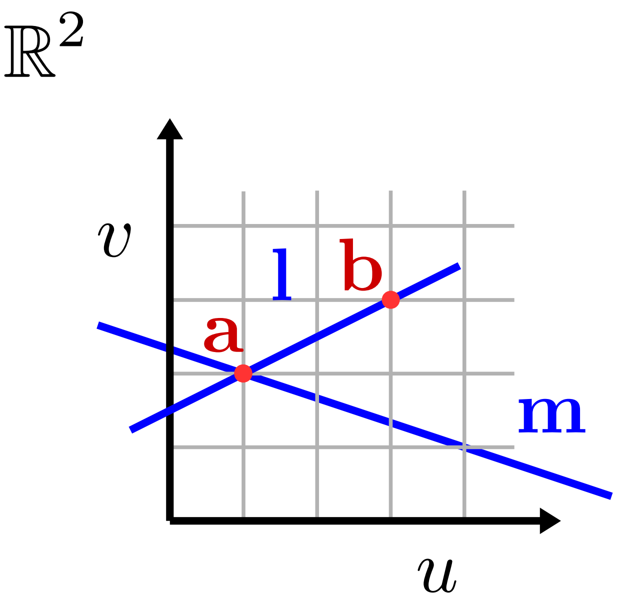
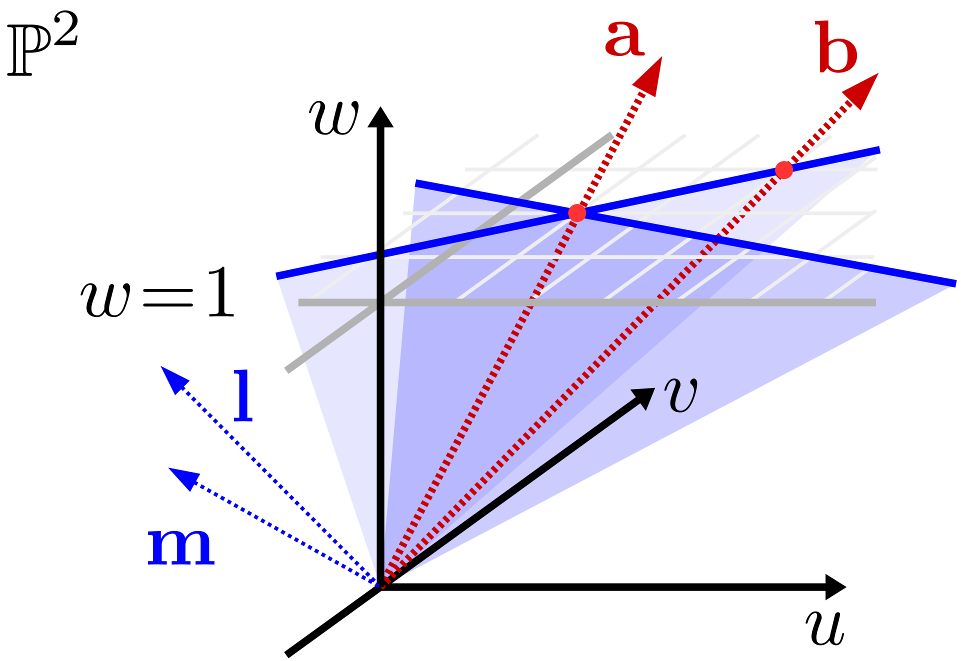
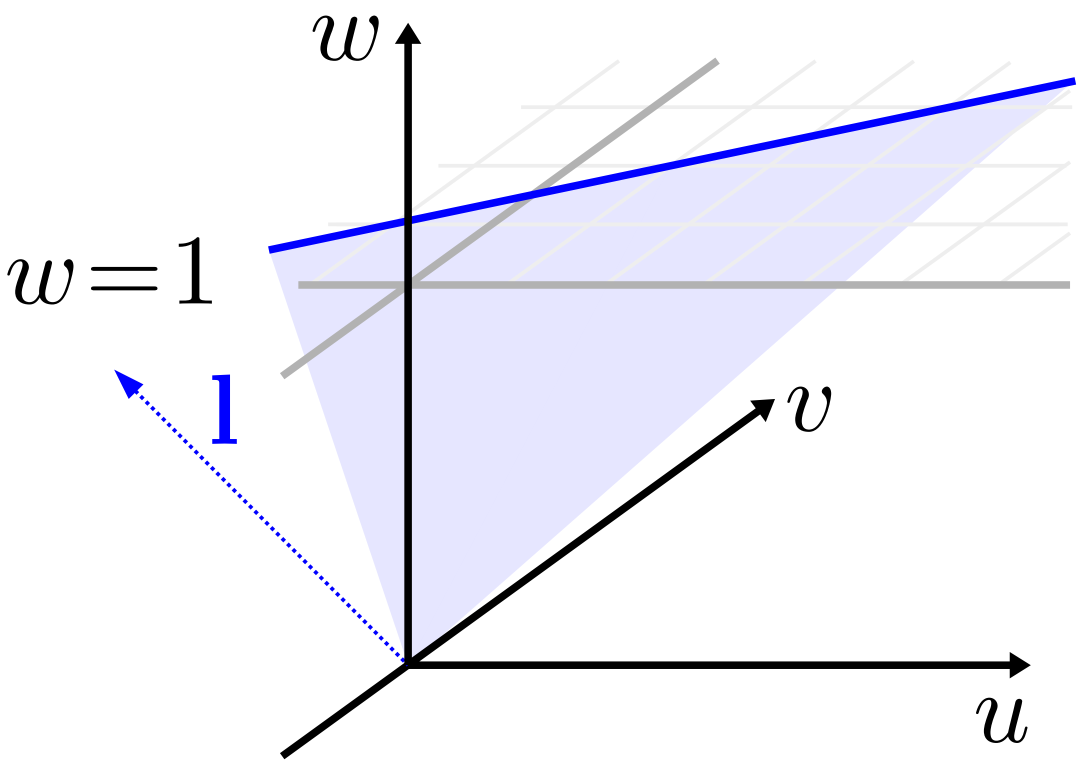
...let's visualize this space...





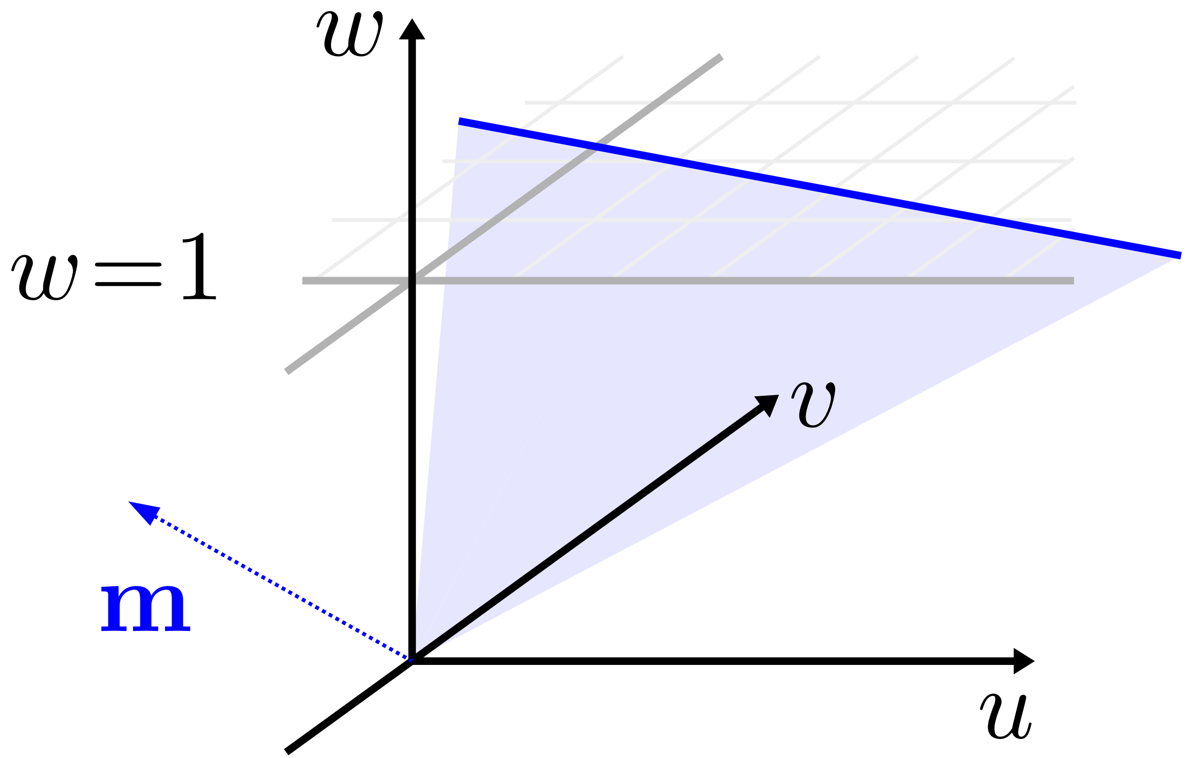
Lines are planes through origin (defined by normal)

$\mathbb{P}^2$



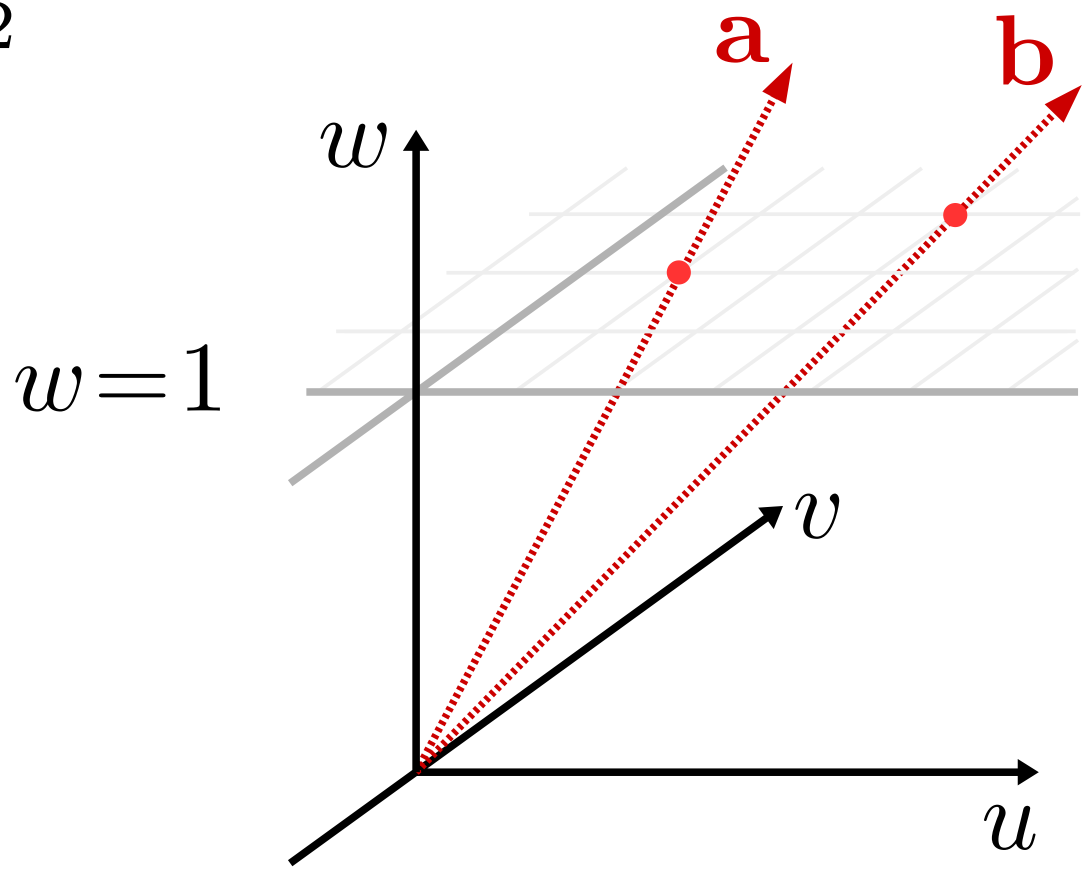
Lines are planes through origin (defined by normal)

$\mathbb{P}^2$

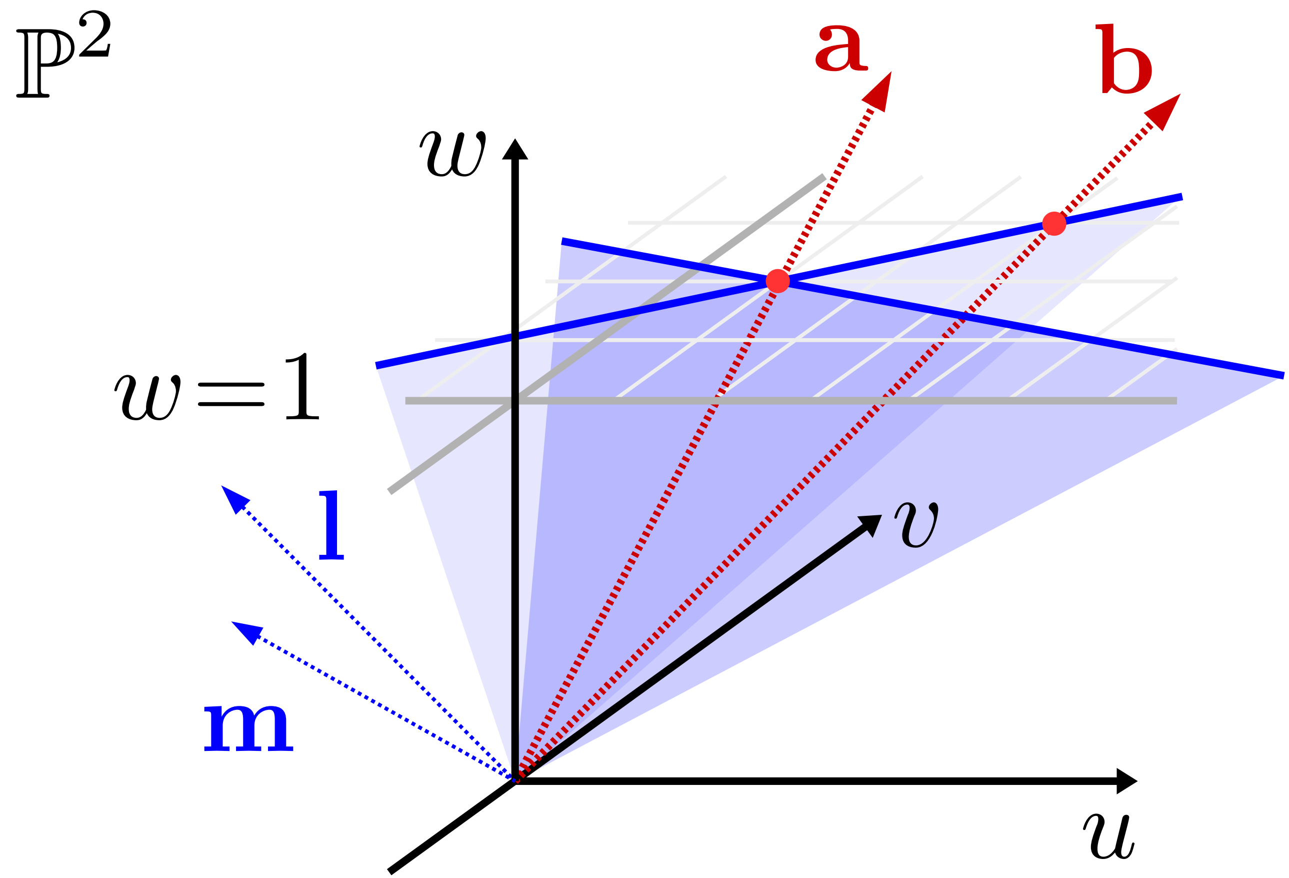


Points are lines through the origin (defined by direction)

$\mathbb{P}^2$



Orthogonality is key!



Infinity

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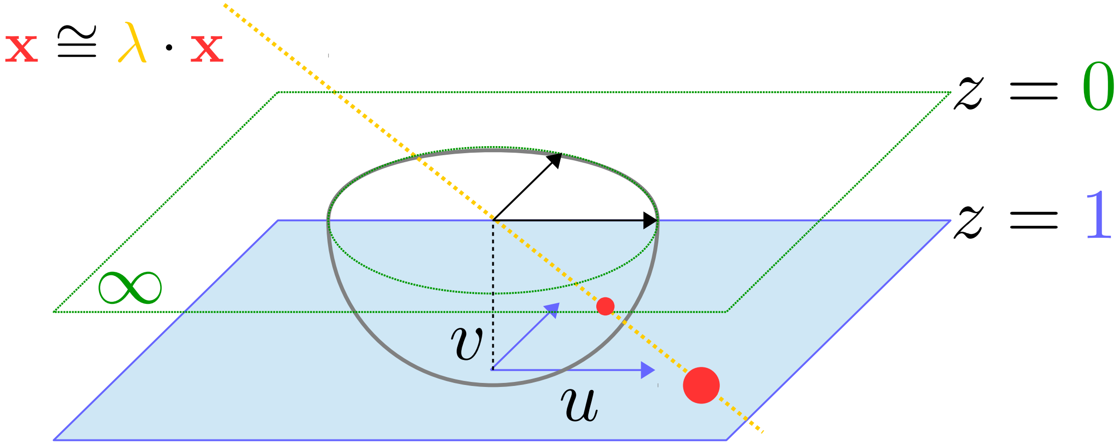
Relation to Spherical Geometry $\mathbb{P}^n \sim S^n \times \mathbb{R}_0^+ \sim S^{(n+1)+}$



$$\mathbf{x} \cong \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} \in \mathbb{P}^2 = \frac{\mathbb{R}^3 \backslash \{0\}}{\mathbb{R} \backslash \{0\}} \sim S^{2+}$$

Lines in Projective Three-Space

- In three-space points are dual to planes
- Join and meet operations (similar to cross product) exist in all dimensions (not covered here: Grassmann Algebra)

2D	Nothing	Point	Line	Plane (E.)	
3D	Nothing	Point	Line	Plane	Space (E.)

Lines in Projective Three-Space

- In three-space points are dual to planes
- Join and meet operations (similar to cross product) exist in all dimensions (not covered here: Grassmann Algebra)

0D	Nothing	Point (Everything)				
1D	Nothing	Point	Line (Everything)			
2D	Nothing	Point	Line	Plane (E.)		
3D	Nothing	Point	Line	Plane	Space (E.)	
4D	Nothing	Point	Line	Plane	Space	4D (E.)
...	...			Plane		...

3D Join and Meet

Line = Point Join Point

```
% line L = point A join point B
function L=join(A, B)
L=[
   A(1)*B(2)-A(2)*B(1);
   A(1)*B(3)-A(3)*B(1);
   A(1)*B(4)-A(4)*B(1);
   A(2)*B(3)-A(3)*B(2);
   A(2)*B(4)-A(4)*B(2);
   A(3)*B(4)-A(4)*B(3)
];
end % function
```

3D Join and Meet

Point = Line Meet Plane

```
% point X = line L meet plane P
function X=pluecker_meet(L, P)
X=[
                - P(2)*L(1) - P(3)*L(2) - P(4)*L(3),
    + P(1)*L(1)             - P(3)*L(4) - P(4)*L(5),
    + P(1)*L(2) + P(2)*L(4)             - P(4)*L(6),
    + P(1)*L(3) + P(2)*L(5) + P(3)*L(6)
];
end % function
```

3D Join and Meet

Plane = Line Join Point

```
% plane P = line L join point X
function P=pluecker_join(L, X)
P=[
                + X(2)*L(6) - X(3)*L(5) + X(4)*L(4);
    - X(1)*L(6)             + X(3)*L(3) - X(4)*L(2);
    + X(1)*L(5) - X(2)*L(3)             + X(4)*L(1);
    - X(1)*L(4) + X(2)*L(2) - X(3)*L(1)
];
end % function
```

3D Join and Meet

Point = Line Meet Plane

```
% point X = line L meet plane P
function X=pluecker_meet(L, P)
X=[
   - P(2)*L(1) - P(3)*L(2) - P(4)*L(3),
   + P(1)*L(1)             - P(3)*L(4) - P(4)*L(5),
   + P(1)*L(2) + P(2)*L(4)             - P(4)*L(6),
   + P(1)*L(3) + P(2)*L(5) + P(3)*L(6)
];
end % function
```

Literature

J. F. Blinn, “A homogeneous formulation for lines in 3 space,” SIGGRAPH Comput. Graph., vol. 11, no. 2, pp. 237–241, Jul. 1977.

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